

Locating Negative Decimals on the Number Line: Insights into the Thinking of Pre-service Primary Teachers

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Abstract

This paper explores misconceptions of the number line which are revealed when pre-service primary teachers locate negative decimals on a number line. Written test responses from 94 pre-service primary teachers provide an initial data source which is supplemented by group responses to worksheets completed during a lesson and individual interviews. Two main misconceptions leading to incorrect placement of negative decimals on a number line are identified. One relates to having separate number 'rays' for positive and negative numbers, which are aligned according to context. The other (with several variations) results from creating the negative part of the number line by amalgamating translated positive intervals. These misconceptions explain a large percentage of wrong answers. The most important implication for education at school, as well as in teacher education, is that the teaching of negative numbers and of the number line must not be confined to integers, as is frequently the case.

Keywords: number line, decimals, pre-service teachers, misconception, cognition, understanding

1 Introduction

One of the most useful representations of number is a number line. The number line is important as it can be used for connecting decimals, fractions, and other types of numbers and for developing a sense of relative size. It is an especially important representation because, unlike many other representations and models used in teaching, the number line plays an important role in mathematics to the most advanced levels. It is used for measuring scales and Cartesian axes, horizontal and vertical, as well as embodying \mathbb{R} , the abstract set of real numbers. The number line is also important for learning because of its epistemic fidelity (Stacey, Helme, Archer, & Condon, 2001) towards characteristics of real numbers such as density; the property that in between two unequal numbers there is another one. A discrete model, such as multi-base arithmetic blocks, cannot faithfully present this property. Hence, the number line is important because it is both a didactical model and also a part of higher mathematics.

The main introductory work on the number line at school tends to be limited to representing positive and negative *integers*. There appears to be an assumption that after introducing school students to the negative integers on a number line, that they can extrapolate their knowledge of other positive numbers (i.e. fractions and decimals) to their corresponding negative numbers. It is also the case that instruction about fractions and decimals concentrates on positive examples, and that negative decimals and fractions receive little attention. As will be noted below, our own teaching has fallen into this trap and we have observed it in many textbooks. This paper reports a detailed examination of the thinking of a sample of pre-service primary teachers when placing decimal numbers on number lines. We provide evidence that many of them exhibit a twisted geography of the negative side of the number line. We identify two major misconceptions that cause high error rates on apparently basic tasks. There are important consequences both for school education and teacher education.

The aim of the paper is to uncover pre-service primary teachers' understanding of the number line, especially the negative region. This is important so that misconceptions, which are also likely in other populations, can be addressed. The fact that pre-service teachers' misconceptions might be passed on to their future students provides a further strong impetus for researching this issue.

In this paper, the word 'decimal' refers to any number written explicitly using a decimal point (or decimal comma), so 2.0 and -0.35 are decimals whereas 2 and 35/100 are not.

2 Literature Review

In this section, we review studies of students' understanding of decimals, and of number lines. Several factors contribute to an inability to locate certain numbers on a number line. For example, someone who is unable to correctly locate a positive decimal correctly on a number line will obviously be unable to locate the corresponding negative decimal. For this reason, we will discuss research about (positive) decimal misconceptions, as well as other research about number lines and negative numbers.

2.1 Difficulties with number lines

Various researchers have documented problems that school students have in locating (positive) decimals on a number line (Kloosterman et al., 2004; Michaelidou, Gagatsis, & Pitta-Pantazi, 2004; Rittle-Johnson, Siegler, & Alibali, 2001; Thomaidis & Czanakis, 2007). Bright, Behr, Post, and Wachsmuth (1988) point out that reading or using a number line requires "an integration of two forms of information, visual and symbolic; this integration does not seem essential with other models" (p. 215). Whereas the number 6, for example, can be represented with 6 objects, or the number $\frac{3}{4}$ can be represented by three shaded quarters of a pie, representing numbers or operations on a number line requires looking at spatial (visual) information (*where is the point?*) and

symbolic information (*where are the symbols for other numbers placed?*). So to place $\frac{3}{4}$, it is necessary to read the positions of other numbers, such as 0 and 1. The number line is therefore a different type of model.

Rittle-Johnson et al. (2001) noticed misconceptions evident in placing positive decimals on number lines (e.g., placing 0.07 as 0.7) as did Thipkong and Davis (1991). These misplacements can be due to inappropriately ignoring zeros or not attending to the size of the marked intervals (e.g., whether the markers indicate tenths or hundredths etc.). McIntosh, Stacey, Tromp, and Lightfoot (2000) reported trials of a computer game (Flying Photographer) to support learning about the size of (positive) decimals. Students are given a position of a virtual animal expressed as a decimal coordinate on a number line and they have to take a virtual photo of the animal at the indicated position when flying a virtual aeroplane over it. The students developed four strategies which assisted in identifying the point on the line given by the decimal coordinate. Use of these strategies did not indicate full place value interpretation of the numbers, but associated knowledge (e.g., a 'sections strategy' placing all numbers of the form 0.3xxxxx in between 0.3 and 0.4 on the basis of the 'tenths' digit). The main finding of relevance to this study is that students often operate successfully on partial knowledge of how to use number lines.

There were few studies (e.g., Peled & Carraher, 2008; Thomaidis & Czanakis, 2007) reporting students' difficulties in locating negative decimals on number lines. Peled and Carraher (2008) discuss the difficulties that students have with both the meaning of negative numbers and the associated operations. For example, a source of difficulty in dealing with negative numbers is the requirement to account for the order relation as well as the magnitude (absolute value) at the same time. Knowing that " -4 is smaller than -2 , although there is more of that quantity in -4 " requires broadening of the meaning of a number (p. 322). Their study provided evidence of students' confusions in enacting operations using borrowing and lending money as representations for negative numbers on the number line. They highlighted the fact that money as a context for teaching negative and positive numbers and their operations is not always accessible for students and even for some adults.

Ordering negative numbers is complex because there are two possible orderings that are supported by thinking about common contexts - the standard ordering and ordering by magnitude (absolute value). Thomaidis and Czanakis (2007) reported that in their sample of 58 sixteen year old students, "most students, when dealing with the left side of the number line, confuse the 'smallest' with the 'greatest'," (p. 177). They put this in perspective by quoting historical findings that both Newton and Euler used terms such as "greatest negative root" for the negative root with the largest magnitude. Thomaidis and Czanakis noted that, for their sample of 16 year olds, (p. 177)

...some of them feel comfortable to stick to the positive semi-axis, and not take into account the negative one. We can hardly avoid paralleling this with many mathematicians' rejection of negative numbers in the past!

2.2 Misconceptions about positive decimals

Extensive studies have documented various misconceptions about decimals in samples of school students (Irwin, 1995; Stacey, 2005; Steinle, 2004). Misconceptions and difficulties with decimals have also been observed in samples of pre-service teachers (Putt, 1995; Irwin, 1995; Stacey, Helme, Steinle, Baturu, Irwin & Bana, 2001; Thipkong & Davis, 1991). These studies indicated that some pre-service teachers shared misconceptions apparent in younger students, although the frequency is different. These misconceptions about the size of decimals are relevant to this paper because the effects of misconceptions with positive decimals need to be identified so they can be separated from difficulties arising from negativity.

A review of studies on pre-service teachers' decimal understandings such as Thipkong and Davis (1991), Irwin (1995), and Putt (1995) supports the findings of Stacey et al. (2001) that pre-service teachers can generally operate as experts on decimals with a few decimal digits (with or without proper understanding of decimals with more digits). If they exhibit misconceptions, these are most commonly from a cluster of misconceptions labeled *Shorter-is-larger*. Shorter-is-larger students consider, for example, 0.9 (the 'shorter' number) to be larger than 0.999 (the 'longer' number) for one of various reasons. For example, they might use *denominator focused thinking* where they consider that 0.999 is 999 thousandths while 0.9 is 9 tenths. Over generalizing the fact that 1 thousandth is less than 1 tenth, this thinking leads to "all numbers with thousandths (i.e. numbers with 3 decimal digits) are smaller than all numbers with tenths (i.e. numbers with 1 decimal digit)". Another reason for the Shorter-is-larger misconception is *reciprocal thinking* which is based on another incorrect connection between decimals and fractions. In this case, the decimal 0.9 reminds them of "chopping up a whole into 9 pieces", whereas 0.999 would involve chopping the whole into many more pieces, each of which would therefore be much smaller. They treat 0.9 like $1/9$, although they do not usually think they are equal. For these and other reasons, some think that a number like 0.999 is very small. The numbers used in the items for this study are generally chosen to avoid the known misconceptions that affect the placement of positive decimals to reduce a confounding variable.

Steinle and Stacey (2004) found that certain ways of thinking that are commonly observed in younger students, are infrequent in older students. In a longitudinal study, Steinle (2004) analyzed what she called 'regression from expertise', and made a strong argument that it provided evidence that many students had learned to follow rules which were later forgotten. Analysis of this regression from expertise resulted in the suggestion that the cumulative effect of many years of school instruction is that *misconceptions are covered over, instead of overcome*. Our concern is that future teachers with misconceptions will pass these onto their students. This study is derived from part of a teaching experiment whose main focus was to develop the desired thorough and correct conceptions of decimals, positive and negative.

3 Method

The data were collected during a research study (Widjaja, 2008) which aimed to improve pre-service teachers' content knowledge and pedagogical content knowledge about decimals and their teaching. The study followed a design research methodology (Brown, 1992) progressing through two cycles a year apart. Each cycle contained a phase of design, then a teaching experiment lasting about 9 class hours followed by retrospective analysis leading to suggestions for improved design. The data reported here is from cycle 2.

3.1 Setting and participants

Data was collected from 94 pre-service primary teachers taught in 3 parallel classes at Sanata Dharma University in Yogyakarta, Indonesia. These pre-service primary teachers (hereafter called students) were undertaking a two-year diploma program run by the primary teacher training department. Classes of pre-service secondary mathematics teachers also took part in the full study but their data is not reported here because very few of these more mathematically able students showed the difficulties discussed in this paper. All the classroom instruction and discussion led by a lecturer, Dr. Susento, was video-recorded and observed by the first author.

The grade 4 curriculum in Indonesia includes statements about negative numbers and their operations. The teaching experiment concentrated almost exclusively on positive decimals. As this paper will show, not including specific teaching about negative decimals in the teaching experiment was an important omission that had been overlooked both in cycle 1 and in the retrospective analysis of cycle 1 undertaken as part of the design research. The consequence for the present study, however, is that data on negative decimals collected at the beginning, middle, and end of the teaching experiment can be aggregated, except in a few cases as noted.

The main method for this study is to undertake very careful analysis of relevant behavior of a large number of students. The student sample has been restricted to the 94 students who were present at all occasions for written data collection. The thinking of these students was inferred from their responses to individual written items and patterns of responses across items. These inferences were then tested against the explanations provided verbally in 27 individual interviews with 14 students. In this paper, explanations from revealing interviews with 2 students are cited. Analyzing the written behavior of many students enables the identification of problems that are common, and hence worthy of attention by teachers and researchers. Supplementing this data source by interviews with a relatively small number of students (in consideration of the time and resources cost of this methodology) enables evidence to be gathered of the thinking that influences student written responses. We therefore have some large population data that ensures relevance of the findings, combined with individual interactions that give insight into student thinking. A further advantage of combining written behavior with

verbal explanation is that human subjects are often not able to fully and accurately explain their own behaviors. These are many instances in the psychological literature (e.g. Elliott & Dolan, 1998) where verbal explanations of choices have been demonstrated not to identify the real reason behind behavior.

3.2 Research Instruments

The data reported includes the complete subset of cycle 2 data involving negative decimals, supplemented by some data on positive decimals as appropriate. This data comes from a multiple-part written question on each of Test 1 and Test 2; group responses to one multiple part question on a worksheet completed in class during the teaching experiment; and from searching the video-recordings of individual interviews and classroom and group discussions for relevant episodes. Checking all the videotapes of all classes revealed, however, that negative decimals only featured on one occasion when students undertook the group discussion on the reported worksheet question (about 10 minutes) and the subsequent 10-minute classroom discussion led by the lecturer to address the many errors that were evident.

The students completed Test 1 before the teaching experiment; the group worksheets during the experiment; and Test 2 immediately after. Interview 1 was carried out shortly after Test 1 was analyzed. This enabled the first author to interview students with interesting responses before the teaching experiment began. Interview 2 was carried out shortly after Test 2 was analyzed. In total there were 27 interviews lasting approximately 30 minutes each conducted with 14 students (some are interviewed both pre and post the teaching).

The relevant items from Test 1, Test 2 and the worksheet are shown in Figure 1, translated into English from the original Indonesian. Note that the original decimal commas have been replaced by decimal points for the purpose of this paper. The Test 2 items were intended to be parallel to the items in Test 1, with one exception. As will be shown, -0.5 (Test 1) can be placed correctly for the wrong reason, so Test 2 used the more revealing placement of -0.35 . The number line on the worksheet question was not marked in tenths, and so we are unable to report positions as accurately as for the test items. In section 4 below, we describe the students' behavior then in section 5 we provide an explanation of the thinking behind the behavior.

4 Results

4.1 Test 1 and Test 2 items

Figure 2 provides the percentage of students who were correct, incorrect or did not answer each test item. For ease of comparison, the three parallel pairs are presented together. Placing positive decimals with non-zero integer part (1.6 and 1.4) were the easiest items with over 90% of the students correct. Placing positive decimals with zero

integer part (0.25 and 0.75) was slightly harder; about 80% correct. The increased difficulty of items containing decimals with a zero integer part is consistent with Australian data reported by Steinle and Stacey (2003) and explained in that paper. Placing negative decimals was much harder; for example, only 57% located -1.2 correctly in Test 1.

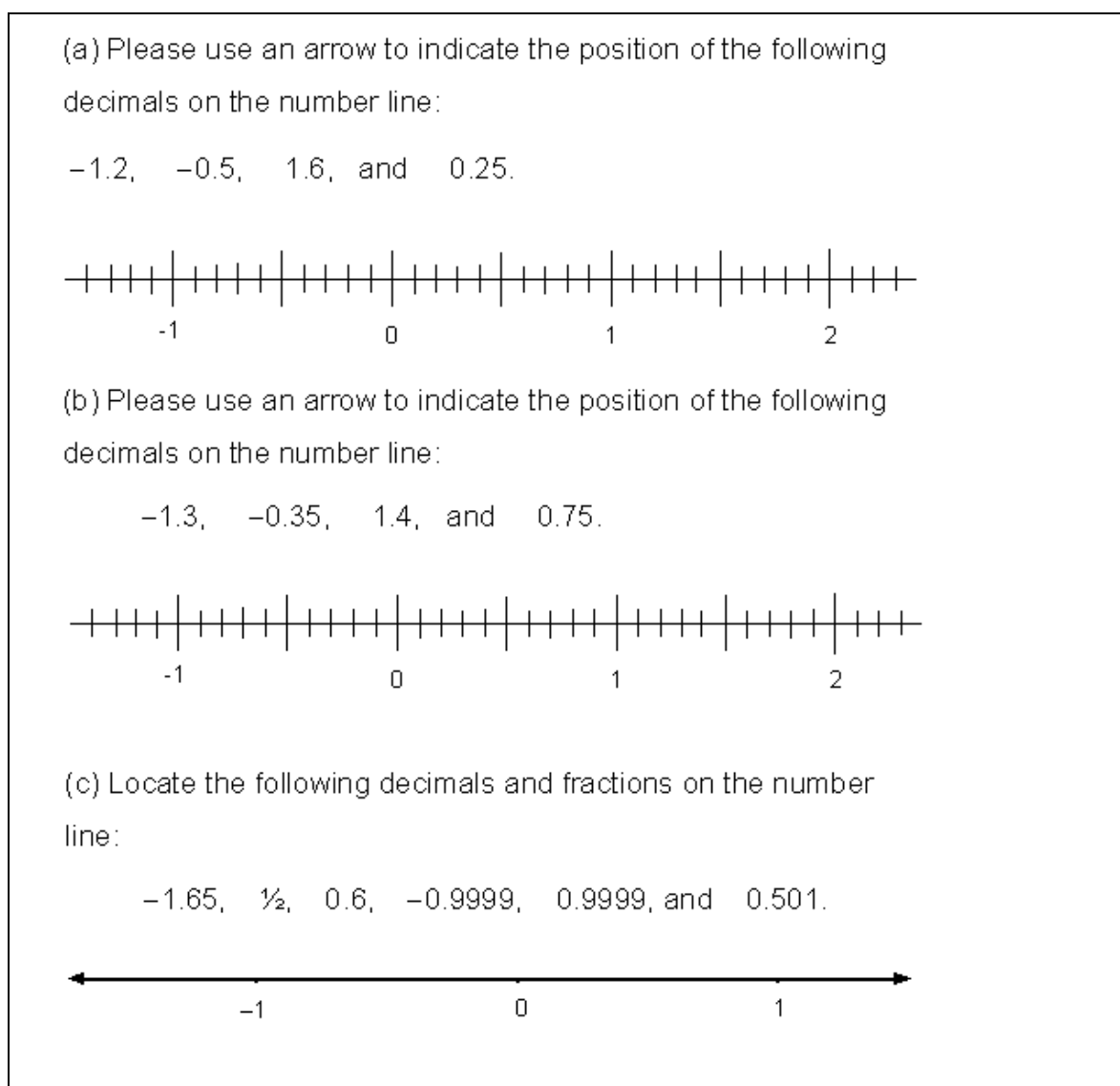


Fig. 1. Relevant items from (a) Test 1, (b) Test 2, and (c) group worksheets.

Figure 2 shows improvement from Test 1 to Test 2. A Chi-squared test showed that the improvement between -1.2 (Test 1) and -1.3 (Test 2) was statistically significant ($\chi^2 = 5.23$, d.f.=1, $p=0.022 < 0.05$). Figure 2 also shows that the number of omissions reduced from Test 1 to Test 2. We assume that both the decrease in number of

incorrect answers and the decrease in omissions are due to the teaching experiment, even though there was very little attention to negative decimals. Students in the post-interviews (e.g. Mariana below) referred to the teaching as changing her thinking.

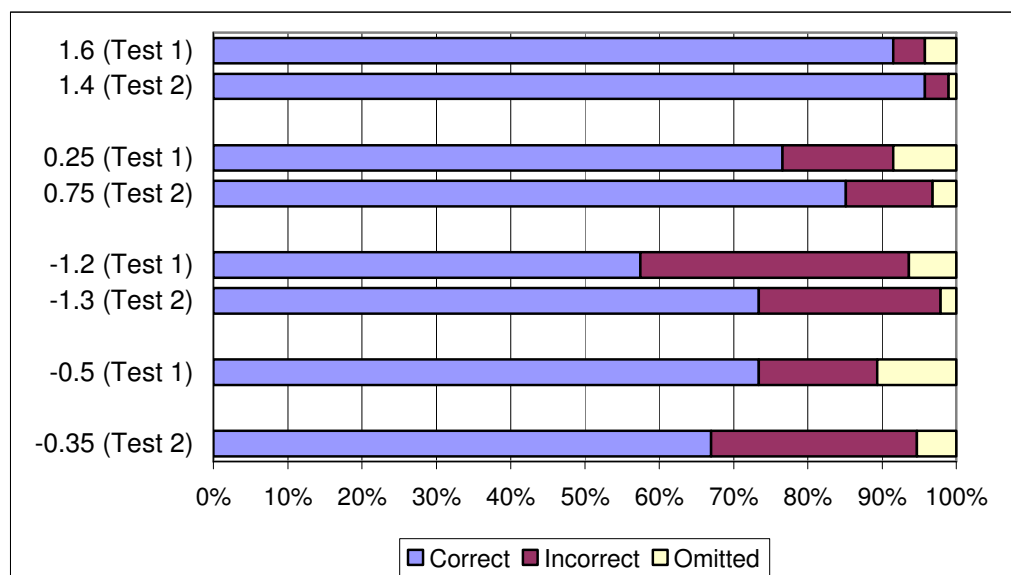


Fig. 2. Distribution of responses to Test 1 and Test 2 items (n=94).

We now analyze the errors on the most difficult items - the four negative decimals. Figure 3 shows the distribution of responses to the placement of -1.2 in Test 1 for all 88 students who provided a response (there were 6 omissions). Similarly, Figures 4, 5 and 6 contain the distributions of responses to the placement of -1.3 (Test 2), -0.5 (Test 1) and -0.35 (Test 2), completed by 92, 84 and 89 students respectively. In these figures, each dot or cross represents a response by a student. Clouds of dots or crosses indicate groups of students who marked the same or an indistinguishably similar point on the number line. Note some students extended the number line to the left and drew additional tick marks. This data is provided graphically to emphasize the relative numbers of students making similar errors.

Figures 3 and 4 show a similar distribution of errors; firstly, almost all the incorrect responses were marked at the same point, and secondly, this incorrect point is located symmetrically about -1 from the correct point. So, 33 of the 34 students who incorrectly placed -1.2 , put it at -0.8 ($= -1+0.2$) and 20 of the 23 students who incorrectly placed -1.3 put it at -0.7 ($= -1+0.3$).

Figures 5 and 6 show a greater spread of errors for placing negative decimals with a zero integer part. In particular, 5 students placed -0.5 very close to zero and 4 students placed -0.35 very close to zero. A few students added an extra tick mark to incorrectly locate -0.5 at -1.5 . Similarly some students extended the number line and added extra

tick marks to locate -0.35 at around -1.65 . The different spread of incorrect placements of -0.35 proves that this item is not parallel to the placing of -0.5 in Test 1.

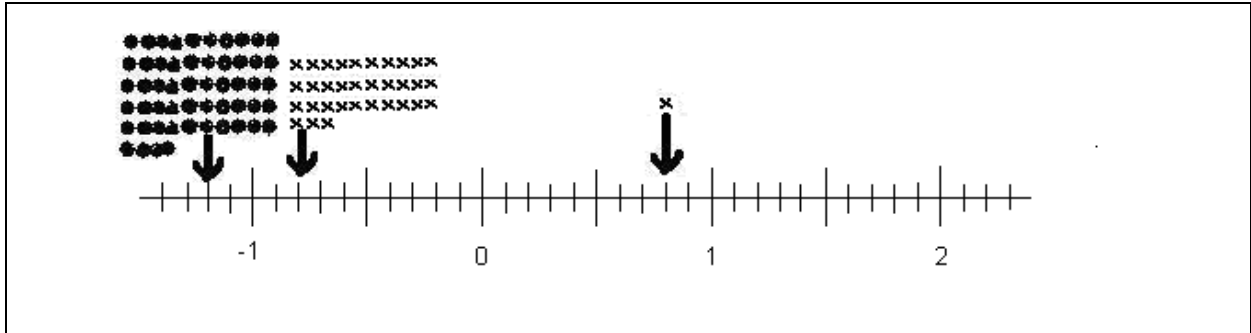


Fig. 3. Graphical distribution of responses to -1.2 in Test 1 for all 88 students who responded. Each student's response is shown as either correct (\bullet) or incorrect (\times).

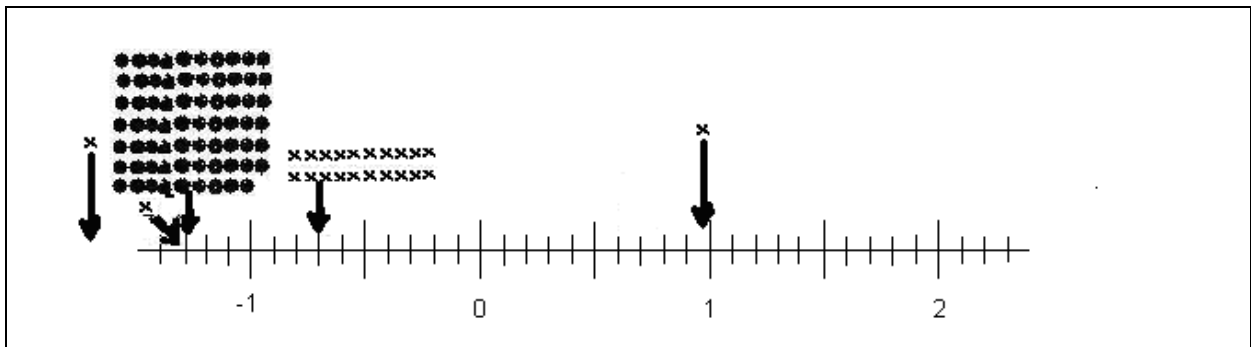


Fig. 4. Graphical distribution of responses to -1.3 in Test 2 for all 92 students who responded. Each student's response is shown as either correct (\bullet) or incorrect (\times).

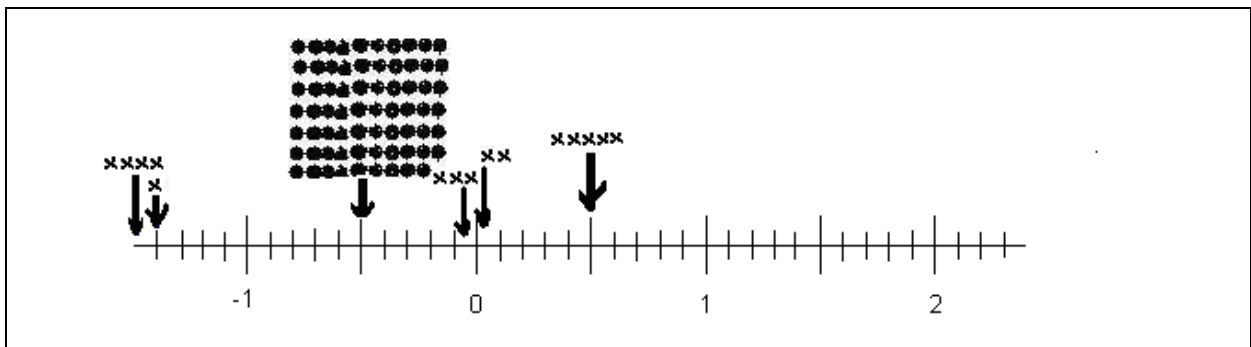


Fig. 5. Graphical distribution of responses to -0.5 in Test 1 for all 84 students who responded. Each student's response is shown as either correct (\bullet) or incorrect (\times).

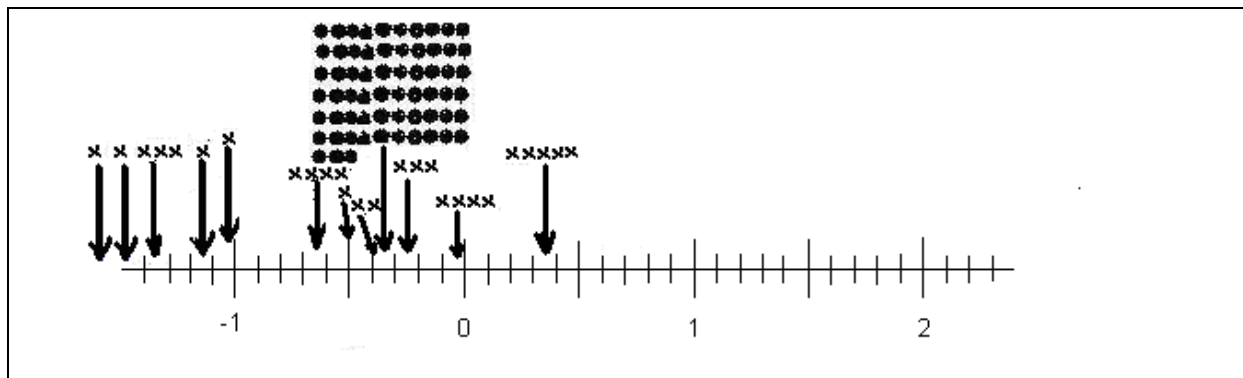


Fig. 6. Graphical distribution of responses to -0.35 in Test 2 for all 89 students who responded. Each student's response is shown as either correct (●) or incorrect (x).

4.2 Group worksheet items

Examination of the worksheets completed as a class exercise by the same students divided into 22 groups provided more evidence of the difficulties in locating negative decimals. Figure 7 provides the results for each of the 6 numbers. The positive decimals (0.6 and 0.9999) and the fraction $1/2$ were placed correctly by 95% of groups. The placement of 0.501 was slightly more difficult (81% correct); two groups incorrectly indicated a point around 0.55 instead of much closer to 0.5. As found in the individual tests, placements of negative decimals were the most difficult with less than 70% of groups correct.

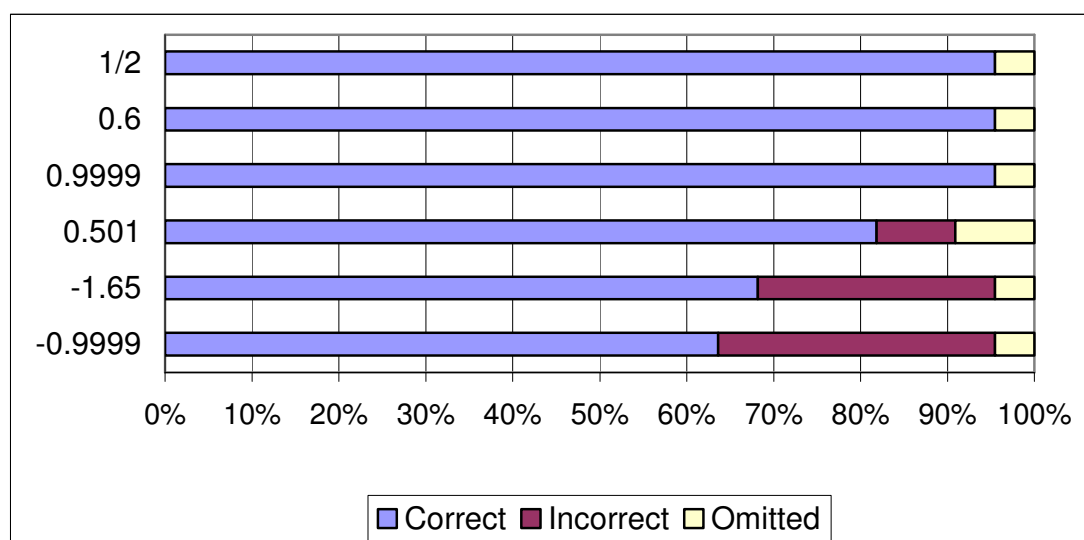


Fig. 7. Distribution of responses to group worksheet items (n=22).

Figures 8 and 9 show the distributions of responses to the placement of the negative decimals, for the 21 groups who answered each item. The only incorrect answer to -0.9999 was to locate it just to the left of 0; 28% of groups did this. In contrast, Figure 8 shows the incorrect responses to -1.65 were more varied.

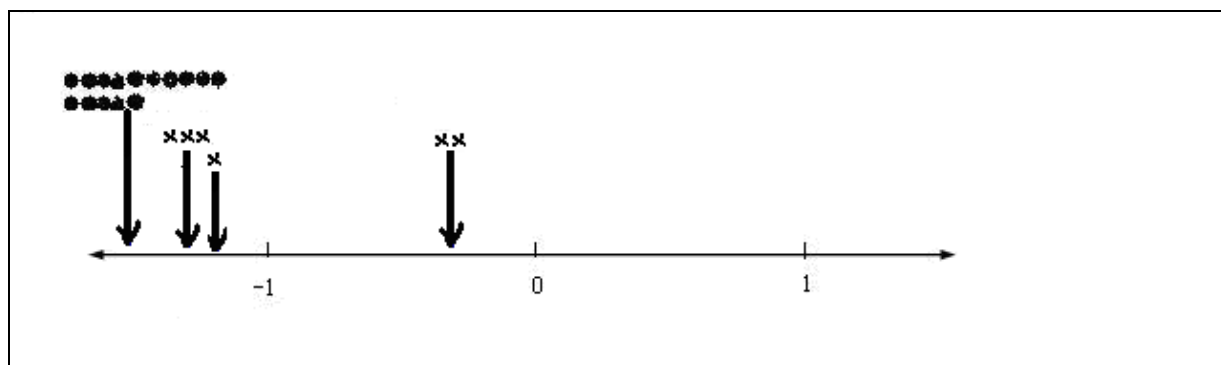


Fig. 8. Graphical distribution of responses to -1.65 on group worksheet for all 21 groups who responded. Each group's response is shown as either correct (●) or incorrect (x).

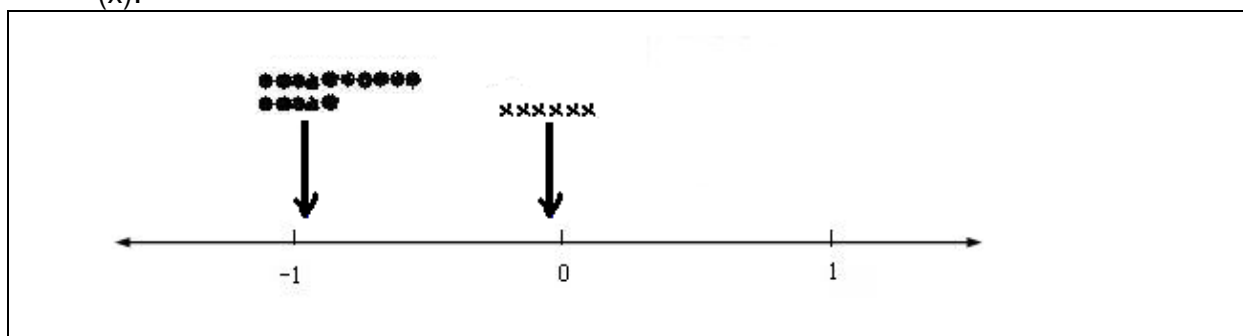


Fig. 9. Graphical distribution of responses to -0.9999 on group worksheet for all 21 groups who responded. Each group's response is shown as either correct (●) or incorrect (x).

The data above provides evidence of difficulties with the placements of negative decimals on number lines. It is important to note that, because of the high success rates for placing positive decimals, these errors cannot be adequately explained by the documented decimal misconceptions such as “longer-is-larger” and “shorter-is-larger” as discussed in Stacey et al. (2001b) and Steinle (2004). It is possible, however, that some of the groups placing -0.9999 very close to zero were influenced by ‘shorter-is-larger’ thinking (which leads them to think 0.9999 is very small) which was prompted because this number has four decimal digits. Positive decimals with only tenths and hundredths (1 or 2 decimal digits) are not so likely to reveal decimal misconceptions because students respond with routinized responses. What we have presented above,

however, is evidence that considerable numbers of students are making similar errors. Are these 'merely' careless errors or are they systematic, predictable errors which indicate underlying misconceptions? In the next section we demonstrate the second alternative.

5 Exploring the thinking behind the errors

Careful analysis of the responses presented above lead to the proposal that there are two misconceptions specific to interpreting the negative part of the number line, which explain nearly all of the incorrect responses observed. The next section describes these misconceptions and their variations, and presents the supporting evidence from interviews and other sources. As with all analyses of misconceptions, students do not always apply these with logical consistency. The intention of the researchers is to reveal the underlying models and sources of thinking and behaviors, rather than models that all students demonstrating the behavior can acknowledge explicitly and apply rigorously.

5.1 Separate negative number ray misconception

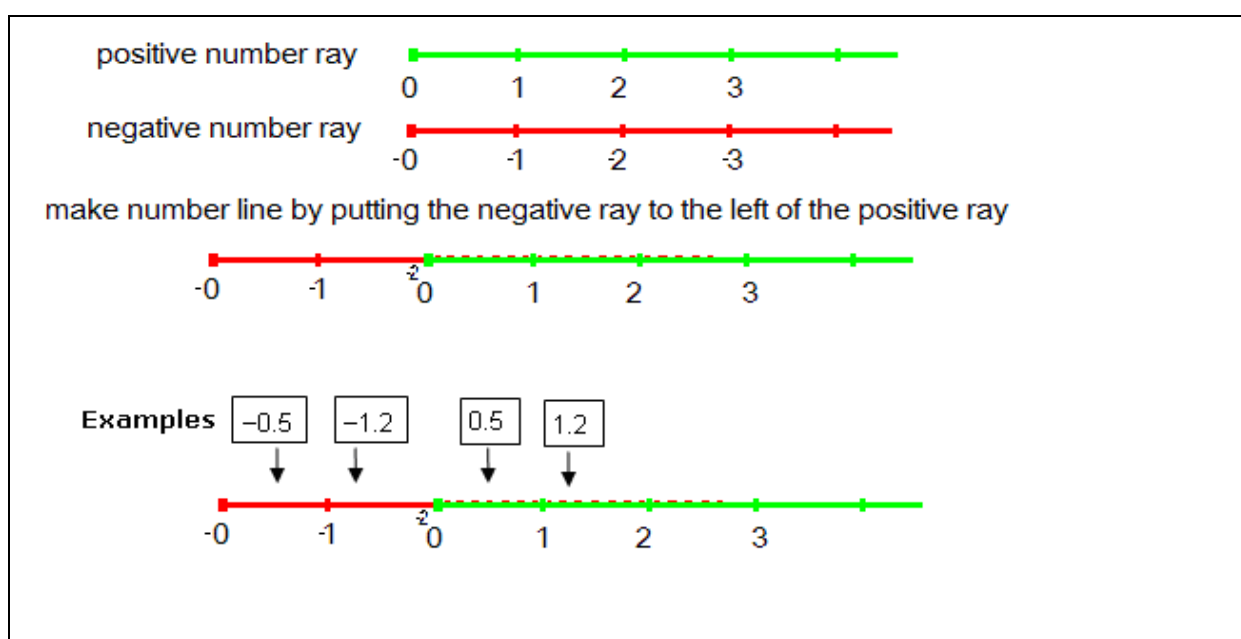


Fig. 10. Separate negative number ray misconception.

A person with this misconception knows where the positive numbers are placed on the number line. We will now refer to the positive 'half' of the number line starting at 0 as the 'positive number ray'. This is the version of the number line that is used in elementary school before negative numbers are studied. Students with this misconception can

correctly place numbers on the positive number ray (excluding the effect of other decimal misconceptions). Corresponding to their 'positive number ray', they create a separate 'negative number ray' which has the same orientation, but all the numbers (often including zero) are negative (see Figure 10). To show the positive and negative numbers together, they then put the number rays together, meeting at a convenient spot or a spot determined by context. This leads to the appearance of a positive zero and a negative zero on the resulting *pseudo-number line*, as indeed happened in an interview described below.

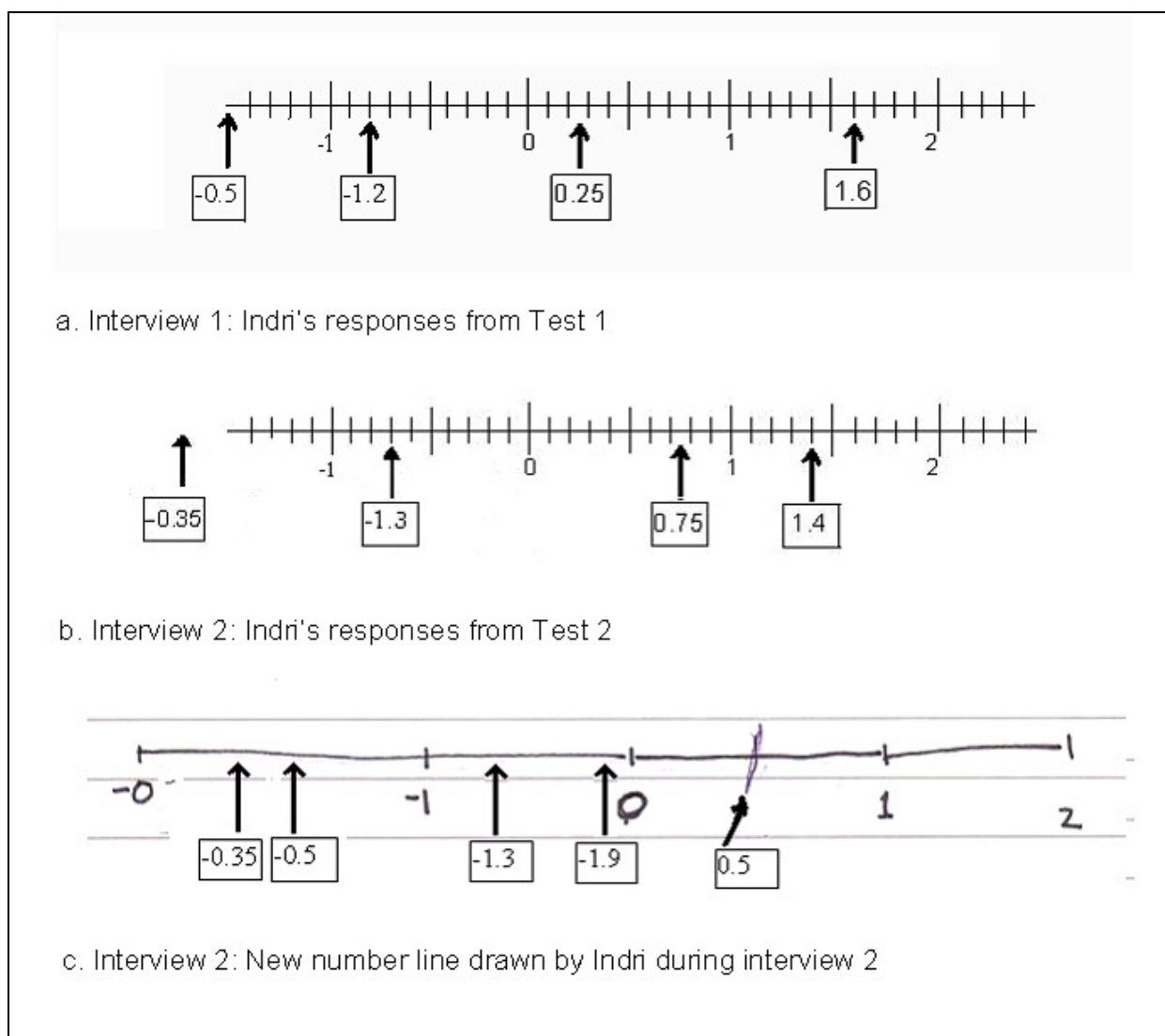


Fig. 11. Indri's placements during interviews (reformatted for clarity).

Interviews with a student Indri revealed this misconception. She participated in two interviews; one after each test. Figure 11 shows her placements of decimals in Tests 1 and 2, which were discussed during Interviews 1 and 2, as well as an extra number line that she drew in Interview 2. These diagrams preserve her placements of numbers but have been reformatted for clarity.

All of Indri's responses were consistent with the separate negative number ray misconception. In Interview 1, Indri was shown her paper from Test 1, where she had correctly located the positive decimals (0.25 and 1.6) and incorrectly located the negative decimals. During this interview, she explained how she had located the two negative decimals (-1.2 and -0.5); see Figure 11a for the resulting number line. Unfortunately the interviewer (the first author) did not probe further into Indri's thinking in Interview 1.

In Test 2, Indri's responses were again consistent with this misconception. During Interview 2 she was shown her paper from Test 2 and asked to explain how she located the four decimals. Her verbal response confirmed her misconception, especially when she suggested the existence of two zeros.

Researcher: Can you explain how you placed minus one point three on the number line?

Indri: Minus one point three is here [pointing to the location of -0.7 for -1.3] but minus zero point three five is here [pointing to the location of -1.65 for -0.35] because minus zero is here [pointing to the location of -2 for -0].

Researcher: So after zero there is minus one and then?

Indri: Then minus zero.

Researcher: According to you, is zero different than minus zero?

Indri: Yes, they are different.

Researcher: After minus zero are there other numbers?

Indri: None.

At this point, Indri was asked to draw a new number line which she correctly marked with 0, 1, 2 but incorrectly marked the left hand end of this number line with -0 (see

Figure 11c), consistent with the misconception.

Researcher: Can you draw them on the number line?

Indri: Here is zero, minus one, and minus zero [marking the location of -2]

Researcher: Where is the 'one'?

Indri: One is here [correctly marking 1].

Researcher: How about minus two? Is there minus two?

Indri: Minus two is here, before minus zero.

Researcher: How did you find that?

Indri: [silence].

Indri has drawn a number line which matches the separate negative number ray misconception. Further questioning, however, reveals that she is unsure about the location of -2 . At this stage, Indri was silent for a while before the interviewer asked the next question on the relation between magnitude of numbers and the position on the number line. As can be seen in the continuation of the transcript, Indri associated negative numbers with debt (owing money) but this did not help her order them on the number line. She was able to articulate that the location of zero was her major point of confusion which resonated with her knowledge of zero as the start of positive (whole) numbers.

Researcher: So according to you, which one is larger, minus one or minus two?

Indri: Minus one.

Researcher: Why?

Indri: Because minus one means you owe one whereas minus two means you owe two.

Researcher: So, on the number line, where is the larger number located, the right or the left side?

Indri: To the left side [pointing to the left direction] ... but no... to the right side

Researcher: So where is minus two?

Indri: In here, I revised my answer [crossing out -0 and putting -2 in the correct location] but I am still confused.

Researcher: Can you tell me what made you confused?

Indri: The zero

Researcher: How exactly are you confused with the location of zero?

Indri: Because I think zero could be here [in the correct location of 0] or it could be here about after minus one [pointing to the real location of -2]

Researcher: Why do you think zero can be in those locations?

Indri: Perhaps because zero is a whole number

Researcher: What do you mean?

Indri: Because after zero, there is one, two, ... zero is always the beginning

Indri's interviews suggest that she perceived that the order of the negative decimals followed the order of positive decimals on the positive number ray as shown in Figure 10. The fact that she thought there was -0 before -1 and that -2 was located just before 0 matched the separate negative number ray misconception. We suspect that Indri would have placed -0 near the left end of any number line that she was given, as long as it contained some negative numbers.

Inspection of Figure 3 indicates that Indri was one of the 33 students who incorrectly placed -1.2 at -0.8 in Test 1; similarly Figure 4 shows that she was one of the 20 students who incorrectly placed -1.3 at -0.7 in Test 2. Can we conclude that all these other students also have the separate negative number ray misconception? No. In the next section we provide evidence that there is another cause for this behavior.

5.2 Translating positive intervals misconception with five variations

A person with the TPI (translating positive intervals) misconception knows the correct location of positive numbers as well as negative *integers* on the number line. The positions of the negative *decimals*, however, are incorrectly found by translating the intervals of positive numbers between integers to the negative region. These students know that 1.2 is to the left of 1.3 (on the positive number ray) and assume that the same relationship holds for negative numbers so that -1.2 is to the left of -1.3 . Effectively, instead of interpreting -1.2 as $(-1 - 0.2)$, they interpret it as $(-1 + 0.2)$ although they are

unlikely to think in this mathematical way, nor would they use the language of translations.

We use the notation $4.x$ to represent decimals like 4.2 and 4.35 rather than describing these decimals in more mathematical terms such as “decimals between 4 and 5”. This focuses our attention on the written symbols in a naïve way. Thus $-6.x$ represents decimals like -6.2 and -6.35 .

The translating positive intervals (TPI) misconception with its five variations is illustrated in Figure 12. In TPI1 (see Figure 12), the $-1.x$ decimals are placed between -2 and -1 ; more generally $-n.x$ decimals are placed between $-(n+1)$ and $-n$. Negative decimals are therefore located between the correct negative integers in TPI1, but their positions within this interval are incorrect (-1.2 appears to the left of -1.3) due to the translation instead of reflection.

The other four variations (TPI2, TPI3, TPI4 and TPI5) result from placing these same translated positive intervals in slightly different positions. In all of these four variations the $-1.x$ decimals are placed between -1 and 0 and the $-2.x$ decimals are placed between -2 and -1 . In general, the $-n.x$ decimals are between $-n$ and $-(n-1)$ with one exception. In Figure 12, we can see that such a placement leaves no interval for the $-0.x$ decimals; four options are offered below:

- TPI2: The $-0.x$ decimals are located in the interval between -1 and 0 , which means some points on the line are associated with two numbers (which may or may not be noticed by students);
- TPI3: The $-0.x$ decimals are squeezed just to the left of 0 ;
- TPI4: The $-0.x$ decimals are squeezed just to the right of 0 ;
- TPI5: The $-0.x$ decimals are located in the interval between 0 and 1 which means some points on the line are associated with two numbers (which may or may not be noticed by students).

Consider the location of -0.5 by students with one of these variations of the TPI misconception. With TPI1 and TPI2, -0.5 will be placed correctly; with TPI3 it will be placed just to the left of 0 ; with TPI4 it will be placed just to the right of 0 ; and with TPI5 it will be placed at 0.5 . Figure 5 shows that all of these variations exist in the data.

Further evidence for the translating positive interval misconception comes from inspection of the worksheet solutions. Figure 8 shows that 4 of the 22 groups placed -1.65 near -1.35 (consistent with TPI1) and 2 groups placed it near -0.35 , which is consistent with all of TPI2, TPI3, TPI4 and TPI5. Furthermore, Figure 9 shows 6 of these groups located -0.9999 just to the left of 0 which is predicted by both TPI1 and TPI3.

Mariana is one student whose test responses indicated that she changed from the separate negative number ray misconception in Test 1 to the translating positive intervals misconception in Test 2. In the interview after Test 2, she explained that she changed her ideas between the tests. The lecturer’s instruction was that a negative number could be located on the number line by finding the position of its positive

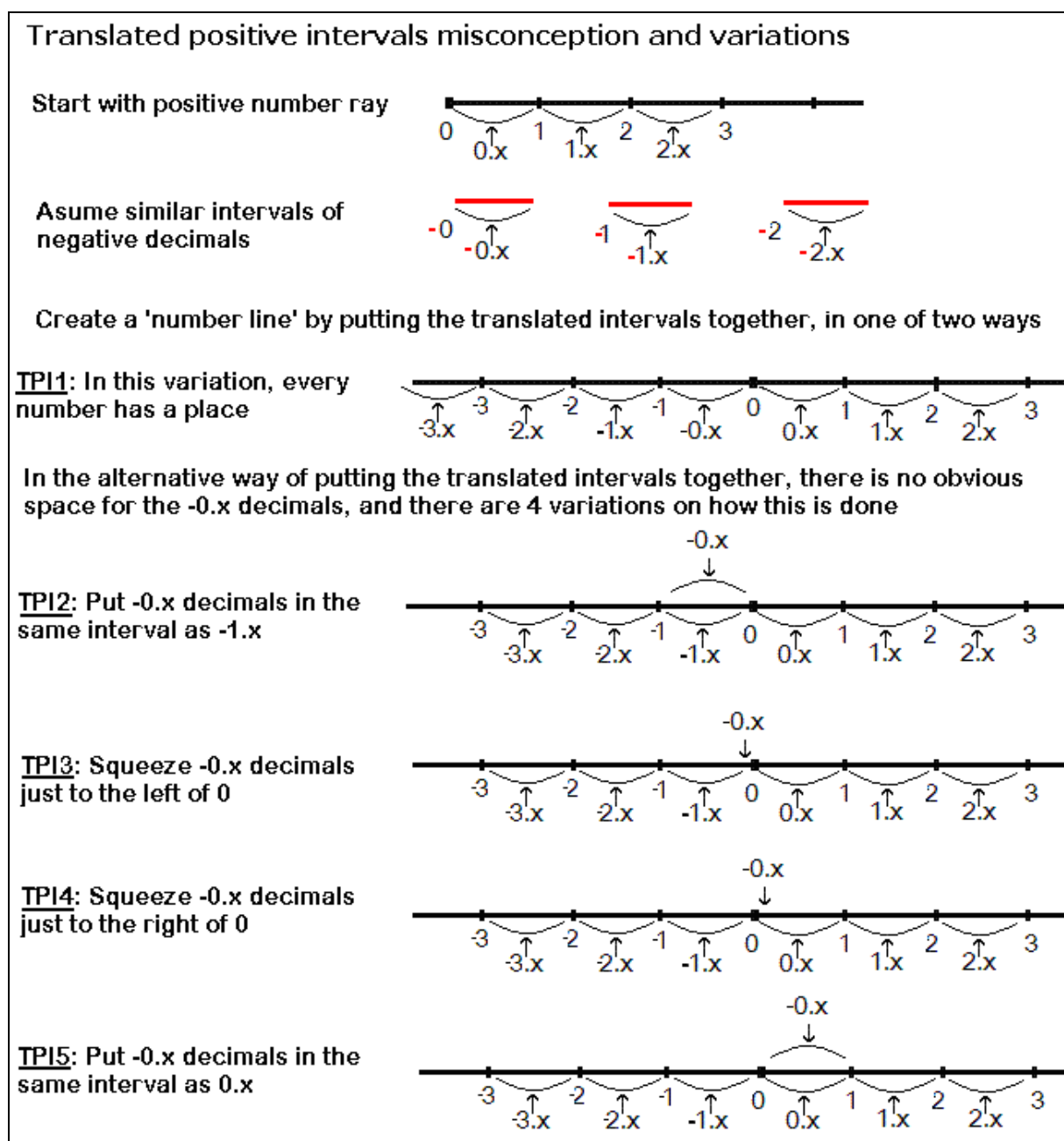


Fig. 12. The TPI misconception showing the five variations.

counterpart and then locating it on the left side of 0, at the same distance from 0, thus moving in the opposite direction. Mariana incorrectly recalled this. She found it difficult (as evident in the interview extract below) to explain why she placed -1.3 at -1.7 (consistent with TPI1).

Researcher: Can you explain to me how to determine the location of minus one point three? [She had located -1.3 at -1.7].

Mariana: In Test 1, I incorrectly located minus one point two so I remembered Mr Susento's lecture that negative numbers are located on the left side of zero.

Researcher: What leads you to locate minus one point three in that position though?

Mariana: Honestly I am still confused when dealing with negative numbers.

The existence of the variations of the TPI misconception that solve the problem of having no space left for the $-0.x$ decimals, provides an example of the 'repair theory' of Brown and Van Lehn (1982) which they developed in their studies of errors in multi-digit subtraction. They explain how systematic errors may be the result of a 'repair' to erroneous procedures as follows. In our case, these repairs are responses not to incorrect procedural knowledge but to incorrect conceptual knowledge.

"The theory is motivated by the belief that when a student has unsuccessfully applied a procedure to a given problem, he or she will attempt a repair. Suppose he or she is missing a fragment (subprocedure) of some correct procedural skill, whether because he or she never learned the subprocedure or maybe forgot it. Because the missing fragment must have had a purpose, attempting to follow the impoverished procedure rigorously will often lead to an impasse...he or she will often be inventive, invoking problem-solving skills in an attempt to repair the impasse and continuing to execute the procedure, albeit in a potentially erroneous way. We believe that many bugs can best be explained as patches derived from repairing a procedure that has encountered an impasse whilst solving a particular problem." (p. 122)

Examination of the responses on all of the tests show that there are students demonstrating each of these misconceptions consistently, although the prevalence of consistent response is less than 10% for TPI2, and less than 5% each for the others. These percentages may have been larger if students had been forced to locate all the decimals, instead of leaving blanks as many did.

6 Discussion and Implications

The results of this paper clearly show that understanding the ordering of negative decimals and placing them on a number line are challenging topics for a significant proportion of the pre-service primary teachers in this study, as documented by their written solutions to tests and workshops and the corpus of the interviews, only 2 of which were reported here.

The study found a low incidence of positive decimal misconceptions in this population, which is consistent with other reports such as Stacey et al. (2001b) on pre-service teachers, but it is important to note that the items used in this study are not designed to highlight them and in any case, the teaching experiment addressed them. Difficulties in scaling, applicable to placing both positive and negative decimals on the number line, were also evident in some wrong answers (e.g., in Figure 6). However, the study has found further evidence for documented difficulties with positive decimals. It supports an unexpected finding from work on positive decimals by Steinle and Stacey (2003) that decimals with zero integer part (e.g., 0.35) are conceptually harder than decimals with a non-zero integer part, and so would benefit from special treatment in teaching.

Difficulties with ordering negative decimals go beyond the difficulties and misconceptions of understanding numeration of positive decimals that dominate the literature. The two misconceptions that have been identified (using a separate negative number ray and translating positive intervals with its five variations) have explained nearly all of the responses to the 6 items. Only in one item (-0.35) do there remain a significant number of unexplained responses, and these may related to difficulties with calculation, or scaling related to having two decimal places. Although there is a high prevalence of all of the responses predicted by the misconceptions, there is not a high prevalence of students consistently following one of the misconceptions proposed. The number of omitted responses is one reason for this. Omission probably indicates uneasiness with other responses, or an unresolved impasse. It is also likely that these misconceptions are for many students, not ideas that are strongly held, but are more a collection of ideas that are drawn upon when a task at hand throws up a challenge.

The results in this paper underscore the need to better prepare pre-service primary teachers for their work and to attend to the teaching of this topic in schools. This is especially the case since the number line is not just as a pedagogical tool, but also an important mathematical object. Teaching about decimals (including our own teaching experiments of both cycle 1 and 2) is concerned almost exclusively on positive decimals. In our experience, teaching about negative numbers is also almost exclusively on integers, including all the work in the Indonesian grade 4 and 5 textbooks and the initial chapters on directed number (revealingly labeled “Integers”) in Australian secondary textbooks. It seems that the textbooks have attempted to make calculation easy by using only integers, but in the process that omitted an important conceptual

development. The misunderstandings revealed in this study demonstrate the need to address the gap of negative fractions and decimals, because their ordering is not intuitive. To get a full appreciation of the real numbers and of the number line, students must learn to work with negative fractions, negative decimals and negative integers as well as the corresponding positive numbers all on the one number line.

The distorted geography on the number lines of many pre-service primary teachers' in this study also highlights the need to provide them with strong visual models of negative numbers and their ordering. The evidence from Indri's interview is that some standard models of negative numbers (e.g. lending and borrowing money) are unhelpful for ordering negative numbers. Distances measured in opposite directions from a central point is perhaps the most obvious visual model. Concrete materials, such as the linear arithmetic blocks (Department of Education and Early Childhood Development of Victoria, 2009; Stacey et al, 2001a; Steinle, Stacey & Chambers, 2006) were used very successfully in the teaching experiment (Widjaja, 2008), and could readily be used for negative decimals, combining place value structure with both length and direction 'east' and 'west' of a fixed origin. It is also necessary to explicitly address the 'paradox' of the two different feasible orderings of negative numbers: the normal ordering and the potential ordering by absolute value which makes so much sense in everyday contexts. Stressing the negative part of the line as a reflection of the positive part may provide strong and helpful visual images. Understanding misconceptions is important because it provides clues for targeting instruction better to learners' needs.

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