Mapping pedagogical opportunities provided by mathematics analysis software

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Abstract

In this paper we propose a taxonomy of the pedagogical opportunities supported by mathematics analysis software: the wide class of software capable of performing the algorithmic processes necessary to carry out routine mathematical procedures from any branch of mathematics. This taxonomy has been structured at three levels from the perspective of the teacher who must think about the subject, their classroom and student learning tasks. It is presented in the form of a 'pedagogical map', a chart to give a visual indication of potential or actual teaching practices. The map draws attention to possibilities of teaching (e.g. in lesson planning or in teacher education) and gives a graphic display of practice (e.g. for research, or for mapping professional growth). It has diverse uses, including reminding teachers and researchers of available options, identifying where teachers need professional development, and being a catalyst for professional discussion.

Technology; mathematics; pedagogical opportunities; computer algebra systems; dynamic geometry; spreadsheets; graphics calculators

MAS: mathematics analysis tools; CAS: computer algebra systems

Introduction

Over the past 30 years it has been shown that mathematical software, housed in calculators or computers, may be used to enrich the teaching of mathematics. In this paper we propose a taxonomy of the pedagogical opportunities that are supported by the use of a wide class of software which we call 'mathematics analysis software'. The terms are defined below. Because we see this taxonomy being of use as a guide to researchers, teacher educators and practitioners, we present the taxonomy in the form of a 'pedagogical map', a chart which gives a visual indication of potential and actual teaching practices.

There are now many different information and communication technologies used in schools, and in mathematics teaching in particular. The paper is specifically concerned with the pedagogical opportunities afforded by 'mathematics analysis software' (MAS). We use this umbrella term to describe computer software or calculators capable of performing the algorithmic processes necessary to carry out routine mathematical procedures from any branch of mathematics, including any or all of arithmetic calculations, symbolic algebra manipulations, statistics calculations and data display, graph plotting, and construction of geometric figures. In a secondary school classroom these tools may be available as computer software (e.g. spreadsheets, function graphers and dynamic geometry packages) or hand held calculators (e.g. four-function, scientific, graphic, computer algebra system (CAS), with or without geometry). MAS are cognitive tools, as described by Zbiek, Heid, Blume & Dick (2007), in that they facilitate the technical dimension of mathematical activity and allow the user to take action on mathematical objects or representations of those objects. Most of these tools are flexible with regard to the menu paths or commands required to complete a process and have many capabilities, which require some effort to learn. Not all of the valuable ICT for a mathematics classroom is 'mathematics analysis software'. The category excludes software (or those parts of software) that is for presentation only (e.g. present problems to students, courseware), for administration (e.g. keep track of students' progress) or as an interface with the real world (e.g. data loggers). Software such as special purpose mathematics applets available on the internet are often very limited MAS and as such may also show some of the opportunities on the pedagogical map. Our pilot versions of the pedagogical map focused on CAS (Pierce & Stacey, 2008) then broadened to encompass other software (Stacey, 2008). The map presented in this paper has been revised in the light of discussion of the pilot versions with both teachers and researchers. In this paper we have chosen to exemplify pedagogical opportunities with some examples of the use of spreadsheets and dynamic geometry environments but many examples refer to the use of CAS, because such software encompasses a wide range of mathematics analysis capabilities. However, the intention is general: to create a taxonomy that applies across a very wide class of mathematics analysis software.

In its role as a taxonomy, the first purpose of the pedagogical map is to organise and summarise the ways in which MAS can be used to pedagogical advantage. It can, for example, be used to draw attention to possibilities of teaching opened up by MAS for:

- teachers planning lessons;

- teacher educators advising on lesson planning and task design;
- professional developers planning to teach teachers to use technology;
- curriculum developers drawing attention to appropriate pedagogy.

The second purpose is as a tool to describe any given praxis of teaching with MAS. For example, one can ‘map’ a particular teacher’s pedagogical use of MAS at one time or over time. This ‘pedagogical map’ of praxis may be useful for:

- researchers analysing one teacher’s use of technology on a single occasion or over time;
- researchers mapping the ‘progress’ of a group, such as new adopters of MAS;
- researchers analysing lessons incorporating the use of MAS;
- teachers reflecting on their own practice;
- professional developers planning for pre-service or in-service programs.

The sections which follow first provide some background on the changing use of MAS, which is seen as having a gradually strengthening orientation to being about learning (pedagogical use), not just about calculating. Second we offer a description and presentation of the ‘pedagogical map for mathematics analysis software’. Since the map is proposed as a taxonomy, it is intended to be a comprehensive summary of pedagogical uses of MAS. There is now a large literature on both the affordances which technology, and MAS in particular, offers for the improvement of teaching mathematics, and in some cases on how such potential may be realised. For recent examples see Heid and Blume (2008a, 2008b). Within the scope of this paper, it is not possible to give a comprehensive review of the relevant literature and hence demonstrate the comprehensive nature of the taxonomy; instead some relevant literature (often from major review articles) is provided within the descriptions of elements of the map. The third section provides illustrations of the use of the map in one context - to analyse the practice of teachers. Finally, comments about the interpretation of the pedagogical map and its uses are provided.

Background: From functional to pedagogical use

The fundamental strength of any MAS lies in its functional capability: it will execute mathematical algorithms quickly and correctly. Learning to use the relevant hardware and software for functional purposes is essential if MAS are to be of any value in the classroom or elsewhere. With few exceptions (e.g. some calculator hardware and dynamic geometry software as described by Laborde & Laborde, 2008), MAS have been designed to assist mathematicians and workplace users of mathematics to save time and reduce manipulation errors: they were not initially designed for classroom purposes. Consequently the most obvious use of any MAS is functional i.e. to produce accurate ‘answers’ quickly.

Since the early introduction of calculators to the classroom the importance of considering both the ‘functional’ and ‘pedagogical’ views and therefore identifying the varied affordances of technology was recognised. Etlinger, writing in 1974 when four function calculators were first available for classrooms, noted that technology could be used functionally (to produce answers) or pedagogically (to improve learning). This distinction will be drawn in this paper. Etlinger’s article recognizes that functional use of the calculator (i.e. for calculating answers) is its most obvious use and he observes that teachers may accept or reject this possibility for student use. However, the intention of this early article is to promote the pedagogical use of calculators. He recognizes that pedagogical opportunities and their actualisation are less evident to teachers and so he gives several interesting examples. For example, he suggests teachers use calculators to help students find patterns. He suggests that teachers can promote deeper mathematical understanding in students if they exploit anomalies such as the rounding errors that at that time resulted in the calculation $(1 \div 3) \times 3 = 0.999999$. Etlinger's discussion also points out that purely functional and purely pedagogical uses of calculators are ends of a spectrum about classroom use of technology, with activities sometimes having elements of both.

The early use of MAS in universities was generally for functional purposes. Wilf (1982) described 'the disc with the college education' (p.4) and began to contemplate how courses such as calculus might be changed if the use of MAS was incorporated. He put on notice the debate regarding the pros and cons of using MAS for educational purposes. Since then the impact of MAS on teaching tertiary mathematics has progressed slowly, most commonly with primarily functional use. Deeper changes to curriculum or pedagogy have been very patchy (Thomas & Holton, 2003). The situation with secondary schools has been different. MAS have arrived in secondary schools later and integration is still slow (Lagrange, Artigue, Laborde & Trouche, 2003). However, whilst their use is still founded in functional capability, the extension from functional to pedagogical use is being explored at all year levels, as illustrated (for example) in Heid and Blume (2008b). It appears to us that where the functional complexity of the tasks is lower, then the role of technology as a pedagogical tool is relatively more important.

While the goals of early MAS creators were to support functional use, more recently they have also been concerned to design software or supplements to other software which may be used in ways that support student learning. We see an example of this in Texas Instrument's Symbolic Math Guide for learning algebraic manipulations. It allows the student to choose options and confronts them with the results of (accurately computed) incorrect or inefficient choices. The calculator has in-built equation solving capability, but this program is designed to help students acquire pen-and-paper equation solving capability. Further examples of design features to support students learning are Maple's symbol palette and user friendly interface; and the development of Geometers' Sketch Pad's sliders and the ability insert and draw over photos or Cabri-3D's 'folding' nets of solids. In addition, a range of hand-held calculators have been developed specifically for the education market. Such changes can promote a shift in balance from functional to pedagogical classroom use of MAS.

The 'pedagogical map' for mathematics analysis software

Functional opportunities underpin pedagogical opportunities

The pedagogical map we have created has a focus on the pedagogical opportunities afforded by MAS: it does not attempt to cover all aspects of the possible impact of technology on school mathematics. By definition, the primary purpose and strength of any MAS is its functional capacity – to be able to carry out routine mathematical procedures (calculating, drawing, solving etc) quickly and accurately. MAS provide improved speed and accuracy for calculations, manipulations and constructions. It provides improved speed in moving between representations (e.g. functions and their graphs) and it also provides improved access to representations that would be very difficult or impossible in a pen-and-paper environment (e.g. 3-D graphs of functions of 2 variables, dynamic diagrams). These functional capabilities of MAS form a foundation for opportunities for three major types of educational change: curriculum change, assessment change or pedagogical change. In other words, MAS supports change to *what* mathematics is taught; to *how* mathematics is assessed; as well as to *how* it is taught and learned. This is illustrated by figure 1. The pedagogical map to be discussed below delves into the box of pedagogical change, but it leaves unopened the boxes for curriculum change and assessment change. Each of these unopened boxes is worthy of separate exploration.

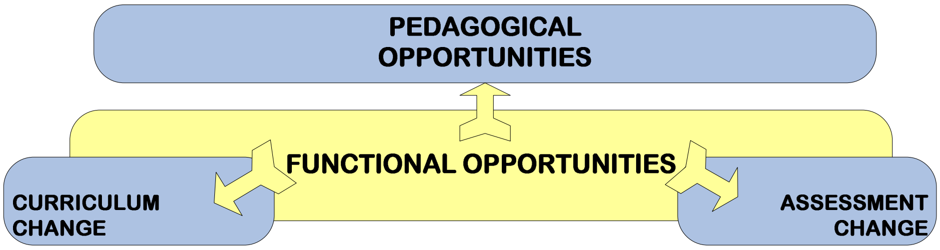

Fig. 1. Mathematics analysis software may impact on pedagogy, curriculum and assessment

The enlarged map, shown in figure 2 zooms in on the landscape of pedagogical opportunities. Knowing that some of the technical work of mathematics may be 'outsourced' to MAS allows teachers contemplate enhanced student learning tasks. In this way the functional opportunities provided by MAS underpin the pedagogical opportunities. This is illustrated on the pedagogical map in figure 2 by the position of the functional opportunities box at the base of the map and the vertical arrows indicating the interaction with pedagogical opportunities.

Pedagogical opportunities

Pedagogical opportunities are openings to change or enhance the way we teach in order to achieve better student learning outcomes. The taxonomy of pedagogical opportunities has been structured at three levels from the perspective of the teacher who must think about:

- the *subject* (i.e. area of mathematics) that they are required to teach
- their *classroom* organisation and
- the *tasks* they will set for their students.

MAS afford opportunities to implement helpful pedagogical practices at each of these three levels and these opportunities are named and described below. In this section we give some simple examples and refer to opportunities noted by various researchers in this field. The categories we have used are broad. They are intended to act as a reminder to possibilities that may enhance the teaching of mathematics. The pedagogical map is shown in figure 2 immediately below and a parallel version, shown in figure 3 below the description of each level of the map, provides a few examples to illustrate each part of the map.

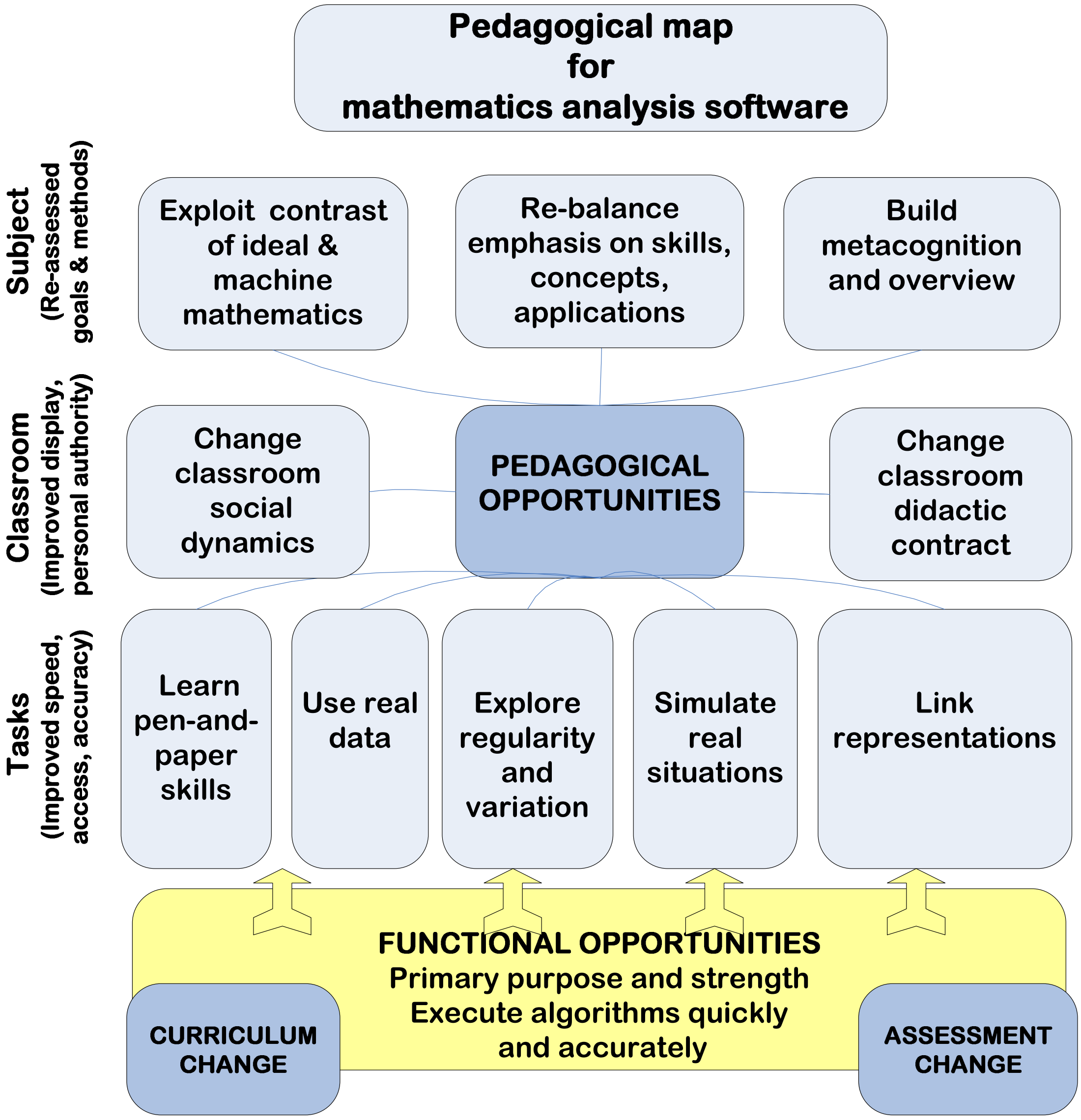

Fig. 2 Pedagogical map for mathematics analysis software

Task level opportunities

The bottom row of the pedagogical map considers changed mathematical tasks. It has five boxes representing five different ways in which MAS affords opportunities for improved teaching and learning tasks (see figure 2 and examples in figure 3). As noted above, and on the left hand side of the pedagogical map, these opportunities depend on the improved speed and accuracy of calculations, manipulations, and constructions, as well as access to previously unavailable mathematical representations. The potential advantages for teaching, opened up by these attributes have been the focus of most research on the use of MAS for teaching. Review articles such as that by Lagrange et al (2003) give many illustrations. We now briefly discuss each of the opportunities in the boxes.

Students' observation and analysis of accurate MAS responses may support their learning of pen-and-paper skills. An undergraduate student quoted by Pierce and Stacey (2001a) exemplifies this opportunity:

> I think it (CAS) actually helps me learn new things because when there are new things that I'm learning, while I'm finding them difficult, I can use DERIVE and go through the steps. With more practice, and seeing DERIVE go through it, I pick it up myself and then I can feel confident doing it myself without a package. (p.11)

At a secondary school level this might involve simple steps such as using CAS to solve equations one step at a time in the home screen rather than using an automatic 'solve' command, or using the Symbolic Math Guide as mentioned above.

Improved speed and accuracy allows access to real world tasks, using real world data where pen-and-paper calculations may be too error prone or time consuming. Such tasks have motivational values as they can engage the interest of students while involving them in valuable mathematical thinking and demonstrating the uses of mathematics. For example students may collect real data of the height of a bouncing ball using a data logger and then use MAS to analyse this data and create a mathematical model.

There are many examples in the literature including Heid and Blume (2008b), where MAS enables students to explore regularity and variation. For example, they may use a slider to vary a parameter to see the effect on the graph of a function, or drag a corner of a triangle in dynamic geometry to see what happens to the sum of the angles, or use a spreadsheet to observe 'what if..." The key to these uses is that MAS provides quick access to many accurate examples. Zbeik et al (2007) note that teachers identified being able to support such investigations as one factor which contributed to successful classroom use of spreadsheets, graphing utilities, geometry programs and databases. Zbeik et al also note that valuing such exploration is consistent with a broadly constructivist philosophy of teaching.

Simulations also rely on speed and accuracy. The random function of a spreadsheet or graphics calculator might be used, for example, to generate histograms for 1000 tosses of two dice or repeated experiments to find out how many cereal packets need to be purchased to collect a full set of novelty cards. Dynamic geometry provides many opportunities for simulations of geometric problems which allow students to explore behaviour or visualise what happens.

The use of MAS to enhance student learning by linking representations has been a prominent theme in the literature. Zbeik et al (2007) draw special attention to the role that cognitive tools, in this case MAS, play in mathematical activity by externalising representations. Theories of multiple representations propose that a key to students' understanding is their ability to link representations and to gain representational fluency, where they can interpret mathematical ideas in distinct representations. In particular the capacity of a graphics calculator to move seamlessly between symbolic, numerical and graphical views of functions has been much studied and innovations using this feature are widely reported. The incorporation of dynamic geometry into these tools adds the possibility of linking to geometric representations as well.

Uses of MAS at the task level in all five categories are regularly reported in the literature. For example, when Heugl (1997) reported on a series of research projects considering the use of MAS in Austrian schools he observed that with CAS new strategies become available and others are easier to use. Tasks set may focus on generality and variation; linking representations; access many correct and varied examples through varying parameters or through 'zooming'; and the use of simulations. Kieran and Yerushalmy (2004) provided an overview of research on the use of MAS, such as spreadsheets and dynamic geometry, for their impact on teaching and learning algebra. They noted how MAS encouraged investigating and generalising through the use of using multiple representations. Spreadsheets, for example, may be used to explore relationships between quantities; support real life applications problems; offer alternative solution methods and the analysis of simulated data, while dynamic geometry may be used to analyse the behaviour of functions by linking geometry with graphs and tables with symbolic representations of functions. Dynamic geometry may also be used to support students' exploration of 'what varies and what doesn't', leading to rich mathematical discussions. Thomas, Monaghan and Pierce (2004) summarised findings from research on the use of CAS for teaching and learning algebra. They report the value of CAS for allowing students easy access to multiple representations and quickly producing accurate results for many examples thereby encouraging experimentation and generalisation.

In summary, the five boxes at the task level of the pedagogical map point to five broad ways in which mathematical tasks can be enhanced by the improved speed, accuracy and access which MAS offer. MAS can support more emphasis on real world tasks to enhance relevance; they can support new ways of learning mathematics through exploration, they can contribute to better understanding, and they can scaffold the learning of pen-and-paper algorithms.

Classroom level opportunities

The middle row of pedagogical opportunities boxes focuses on changes to the inter-personal dimension of the classroom (see figures 2 and 3). When using the word 'change' we are thinking of opportunities that contrast with traditional classrooms in which the teacher is considered to be the sole authority and students work mostly on their own. As noted on the left hand side of the diagram, we consider that these opportunities arise from the improved way in which mathematical working and results can be displayed and shared in the classroom, and from the way in which mathematical authority changes. The literature contains many reports of changes at the classroom level, again often supporting a constructivist approach to the classroom. Exploration using MAS supports the construction of meaning by students linking action and mathematical reflection (Lagrange et al 2003). We have separated these classroom changes into mainly social and mainly cognitive aspects related to the didactic contract.

MAS technology in the classroom introduces an 'authority' other than the teacher, and students may gain a new sense of personal authority. As a consequence teachers' and students' expectation may change and students may be empowered to take greater control over their own learning, resulting in a change to the didactic contract. Along with other authors, Kieran (2007) notes the changing roles for both students and teachers when technology is integrated into the mathematics classroom: Teachers and students may take the roles of consultant and fellow investigator. Students may be expected to be more active participants in their learning rather than relying on the teacher to deliver knowledge. Students may learn how to use MAS to support their own mathematical explorations and to communicate their finding to the teacher and the class. In response to what Artigue (2001) describes as an ‘explosion of methods’ available when using MAS, teachers must decide whether to allow students a range of options or to privilege particular approaches and restrict others. Different didactic contracts, differing in dimensions such as student choice of method, will be appropriate for different teachers and different student cohorts. Kendal and Stacey (2001) describe one such situation.

The availability of MAS may also be a catalyst for change in the social dynamics of the classroom with students working collaboratively and discussing mathematics. The review article by Zbiek et al (2007) draws attention to the powerful role played by cognitive tools (such as MAS) in allowing students to externalise their internal mental representations and so share and discuss their thinking with other students and the teacher. Not only do improved displays for the output of MAS (for example via a smart board or data projector or a shared computer screen) support such collaborative work, novel to those who have experienced mathematics only as a solo pursuit, but the MAS takes the role of a discourse participant (Kieran, 2007).

Heugl (1997) reported that in Austrian classrooms changes were noted in both social dynamics and the didactic contract as lessons moved from being teacher-oriented to student-centred; from whole class work to pair and single work; from students receiving information to being expected to actively produce knowledge through mathematical explorations. Pierce and Stacey (2001b) also reported on undergraduate students’ use of CAS (DERIVE) and its role as in promoting positive learning strategies including changed didactic contract and changed classroom dynamics as students turned to CAS as an available authority. Students worked collaboratively in small groups and they ‘included’ the CAS in these as a helpful friend. They ‘consulted’ the computer as they negotiated mathematical meaning.

Teachers must make careful decisions about how they wish to allow the presence of technology to impact on classroom organisation and dynamics. The guidance of the teacher is vital in using MAS to support learning but the availability of technology opens up opportunities that face the teacher with more choices. If the teacher desires and permits, this may lead to changes in class expectations and organisation.

Subject level opportunities

At the top of the map we have three boxes that note opportunities for technology to support new or changed goals or teaching methods for a mathematics course as a whole (see figures 2 and 3) and new understandings of mathematics as a field of human endeavour. New teaching schemes may be designed to promote deeper learning.

The box on the left notes that the teacher may deliberately capitalise on the constraints, anomalies or limitations of technology to provoke students' mathematical thinking, as they encounter the real and apparent differences between machine and ideal mathematics. Fundamental aspects of doing mathematics are brought to the fore by this focus on the 'mathematical fidelity' of the MAS i.e. "the faithfulness of the tool in reflecting the mathematical properties, conventions, and behaviours (as would be understood or expected by the mathematical community)" (Zbiek et al, 2007, p1173). As noted above, Etlinger (1974) recommended using unexpected rounding to pedagogical advantage. Modern CAS still provide surprises, such as in the pixilation of function graphs, or in the way in which complex number mathematics impinges onto apparently real number mathematics, or the way in which different brands of dynamic geometry behave in different ways and none of them are simply pen-and-paper geometry put inside a computer (Zbiek et al, 2007). Unexpected mathematical results may be distracting and disheartening, or they be used as opportunities for rich mathematical discussion. Such events may be seen as interruptions to a teaching sequence or they can be deliberately generated and harnessed by the teacher to promote discussions of issues that highlight aspects of pen-and-paper or in-head mathematics in contrast to machine mathematics.

The box in the centre notes that technology may be used to alter the balance in teaching between skills, concepts and applications. A shift away from mathematics overly dominated by practice of skills has been the recurring goal. An early instance is Heid (1988), who investigated the pedagogical use of CAS to alter the balance between concepts and skills in a calculus course. She monitored two groups of students and showed that those students, who first focused on learning concepts by outsourcing procedures to a CAS, then followed this by learning the related by-hand mathematical skills demonstrated greater understanding and equivalent competence in pen-and-paper skills as those students who worked entirely by hand. Heid (1988) followed Pea's (1985) distinction and saw this as a capacity of technology to reorganise rather than just to amplify cognitive activity. In geometry, Vincent (2003) used dynamic investigations to act as a catalyst for more abstract geometric reasoning. Its use shifted the balance in her geometry class away from learning facts and supported students' argumentation, helping them to connect conjecturing and proving.

The box on the right notes that instead of starting with detail, teachers may choose to approach topics through different entry points e.g. starting with an overview or real-world motivating application, using technology to generate results, and then going back to look at details. The teaching reported by Garner (2003) and discussed below is one example. Thomas, Monaghan and Pierce (2004) note that higher level thinking may be supported by the use of CAS if teachers take the opportunity to use these devices to support tasks which encourage meta-cognition and overview, for example, to prompt students' development of algebraic insight (Pierce & Stacey, 2004a, 2004b) and their consideration of the issue of how best to communicate mathematics (Ball & Stacey, 2003).

The sections above have outlined the pedagogical opportunities of CAS, grouping them at three levels, and describing broad categories at each level. At each of these levels, the pedagogical opportunities are underpinned by functional capability of the MAS, but it is the teachers' choice on which of these opportunities (if any) will be actualised in a classroom. Figure 3 below provides very brief descriptions of examples of the possible attributes of tasks or teaching that could access the pedagogical opportunity named in the corresponding box in figure 2 above. The next sections will provide teaching examples.

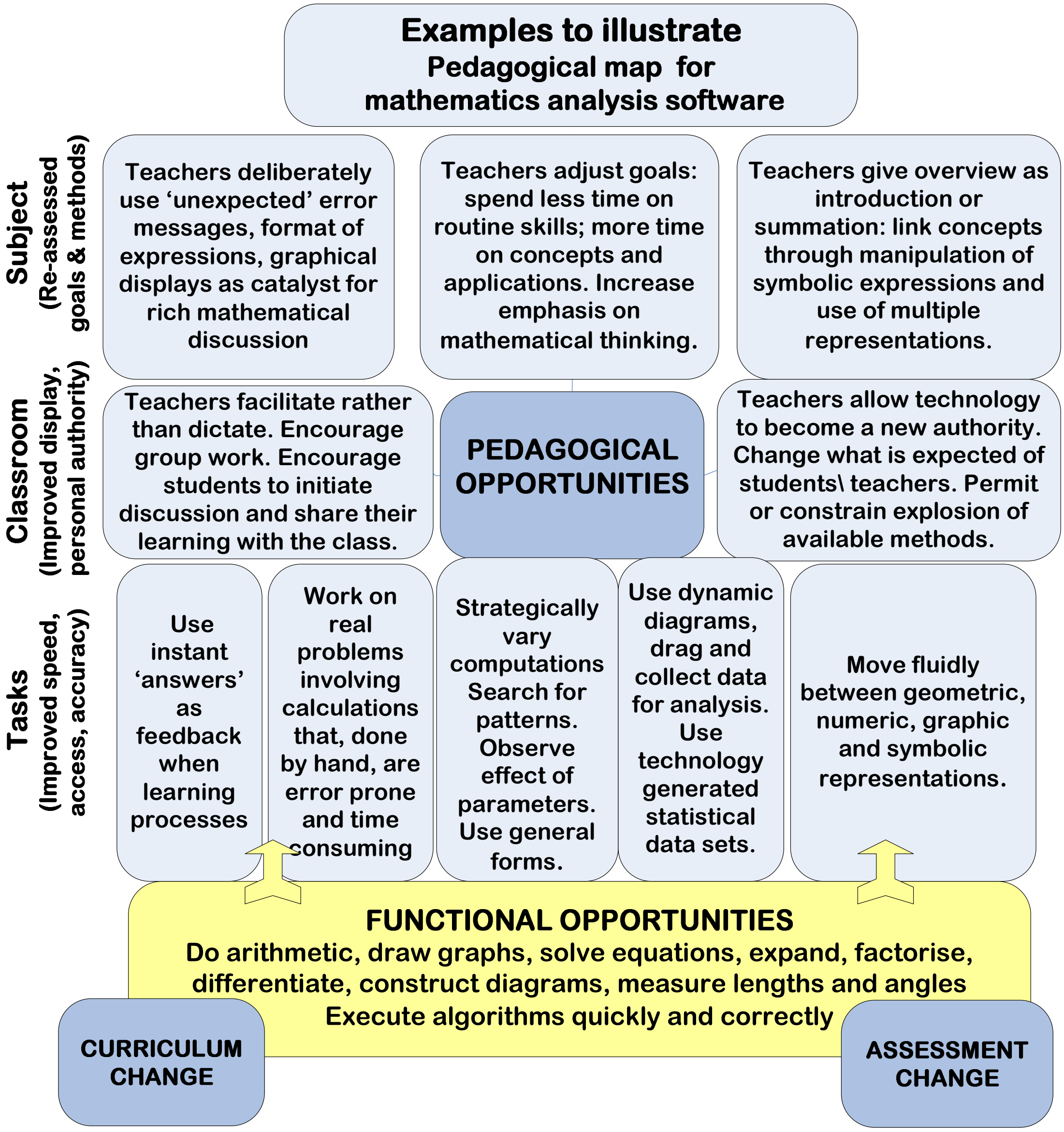

Fig. 3. Examples to illustrate pedagogical map for mathematics analysis software

Using the pedagogical map to describe teaching practices

We listed in the introduction several different uses for the pedagogical map. We envisage it having value for researchers, for teachers and for professional development of teachers. Some of these uses arise from the map's capacity to summarise teaching practice, illustrated below, and some from the maps' capacity to describe or provoke features of lessons. In this paper we have chosen only to demonstrate the one such use of the pedagogical map: to illustrate and so highlight a variety of teaching practices.

In this section we offer mappings of the teaching practices of five teachers, principally using CAS with some use of dynamic geometry in some cases. All five teachers were very experienced and successful. In these cases, the maps do not illustrate the totality of these teachers' use of MAS across all classes and topics but rather their use in the one context we observed and discussed with them. (In other circumstances, teachers may choose to map their own practice more comprehensively, across different teaching assignments.) The maps also concentrate on pedagogical use and not functional use, and hence do not show the type of the functional use engaged in.

MAS supporting real world applications

Many early adopters of CAS for teaching concentrated on the 'functional' opportunities CAS provides. Barney (Yearwood & Glover, 1995), whose teaching is mapped in figure 4, wished to increase the relevance of mathematics for first year undergraduate engineering students. He saw value in the students' learning to use CAS to supplement their pen-and-paper work because the increased speed and accuracy of calculation would allow the students access to real world 'messy' problems. Except for the time spent using technology instead of pen-and-paper, this use of CAS did not disturb the usual pattern of teaching but it did extend the range of problems by adding more application tasks, thus supporting a re-balancing of the goals of the course. Figure 4 maps this pattern of use in teaching, by shading the boxes for functional opportunities, using real data and for re-balancing the emphasis of the subject. In this teaching assignment, Barney was not concerned to use CAS to change the classroom dynamics or the didactic contract, nor to change the nature of the tasks except to include real world data.

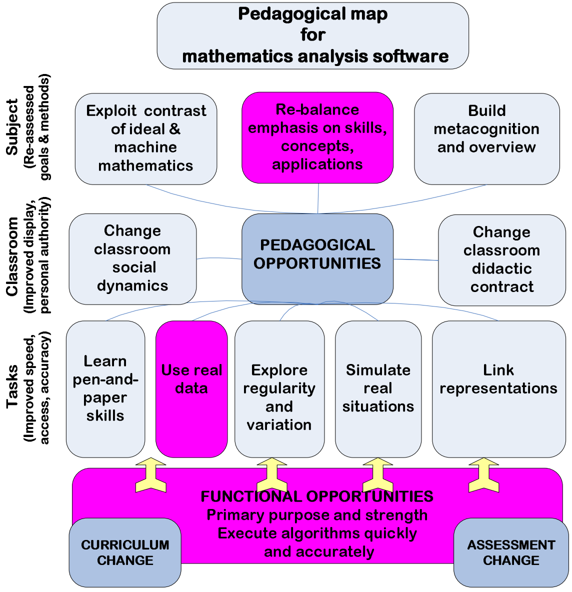

Fig. 4 Barney's use of MAS

MAS supporting speed and accuracy for examinations

Lucy, whose teaching is mapped in figure 5, was a senior secondary mathematics teacher who volunteered for an experimental program where students were allowed to use CAS in the final school examinations (Ball and Stacey, 2007). Observations and self-reports of teaching were made over most of a school year. She believed that her students would learn concepts best by working in pen-and-paper mode but that they would be advantaged in their examinations by using CAS in a functional manner for greater speed and accuracy.

This teacher also valued the pedagogical use of CAS for scaffolding pen-and-paper skills (e.g. by checking pen-and-paper work when students were learning calculus techniques) and allowing easy interchange between graphical and symbolic representations. Her usual teaching practice made regular use of graphical interpretations of functions, and this was greatly assisted by technology. This teacher remained the source of intellectual authority in the classroom. Her pattern of teaching remained fundamentally unchanged as she taught pen-and-paper first, and recommended that students practice first pen-and-paper, and then later use CAS. She often discussed with students whether a particular question would be done most efficiently by CAS or with pen-and-paper.

This excellent teacher made classroom choices that exhibited only 2 of the pedagogical uses of CAS. Figure 5 therefore shows 3 shaded boxes (functional opportunities and 2 others). The case of Lucy highlights that the practice that is judged to suit a given teaching assignment well will not necessarily involve shading in many boxes of the pedagogical map. On the other hand, we contend that it is to a teacher's advantage to have the ability to exploit each of the pedagogical opportunities as the need arises, even though not every opportunity is taken in every class or with all students.

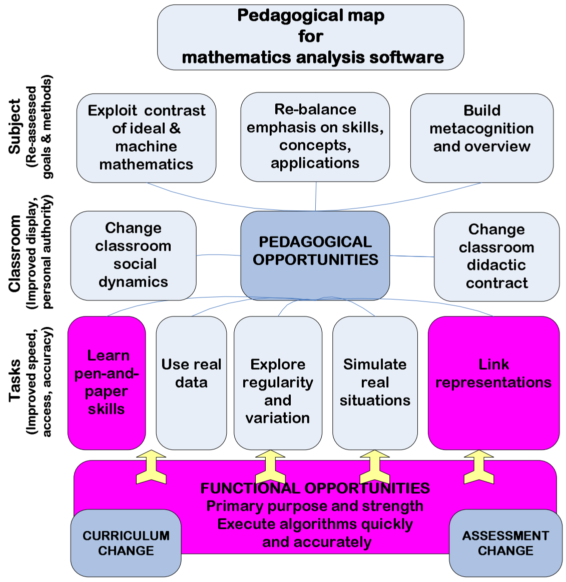

Fig. 5 Lucy's use of MAS

MAS supporting positive learning strategies

Sue, whose teaching is mapped in figure 6, was also teaching senior secondary mathematics and preparing students for the same CAS-permitted examinations. She took the introduction of CAS to the classroom as an opportunity to reflect on, and radically change, her approach to teaching mathematics. In Garner (2003), she described her new approach as starting from the "ends and sides of a topic" rather than at the beginning, as set out in the textbook. Often, with the help of CAS, Sue would now start a new topic with an interesting application or example that the students would not 'normally' have met until the end of the topic. Her intention was to present an overview of what they were to learn and why. The class would then revisit the detail with a mix of pen-and-paper and by-CAS exercises, teacher exposition and sharing the results of students' exploration. This is indicated by shading the top right box.

In this teacher's classroom, both the didactic contract and social dynamic changed. Students were now expected to carry out their own explorations making their own conjectures, generalizations and discoveries rather than relying on the teacher to provide such information. Students were encouraged to share their findings. This included students' projecting their calculator work for the rest of the class to see and publicly explaining their mathematical strategies and CAS procedures.

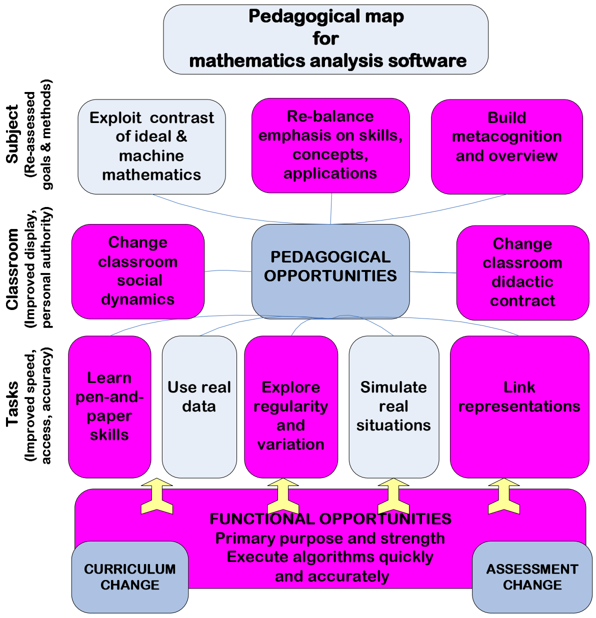

Fig. 6 Sue's use of MAS

MAS supporting positive attitudes

Joe, whose teaching is mapped in figure 7, was teaching 14-15 year olds and is amongst the teachers reported by Pierce & Stacey (2006). He was especially concerned to improve his students' attitude towards mathematics as a step towards improving their intellectual engagement. Joe used technology to compensate for some students' weak pen-and-paper skills and to provide ready access to multiple representations. He believed that this support would encourage his students and increase their confidence by facilitating the experience of successfully solving problems. Joe also aimed to enhance students' interest in mathematics by drawing on real world contexts associated with pleasurable activities from outside school. Sometimes he provided simulated data which was 'cleaned' to highlight important features and sometimes the students collected and worked with 'messy' real data. This teacher cultivated a new sense of cooperation in the classroom with students working in pairs or groups then sharing their results with the whole class. Using technology facilitated this sharing. Students had some results to share, and they had more confidence in results obtained with technology than unassisted, and so public embarrassment was less a threat. As Dickens' (1854) character Sissy in "Hard Times" said half crying, "*You don't know what a stupid girl I am. All through school hours I make mistakes. Mr. and Mrs. M^{c}Choakumchild call me up, over and over again, regularly to make mistakes. I can't help them. They seem to come natural to me*." Joe was keen that his students did not feel like Sissy.

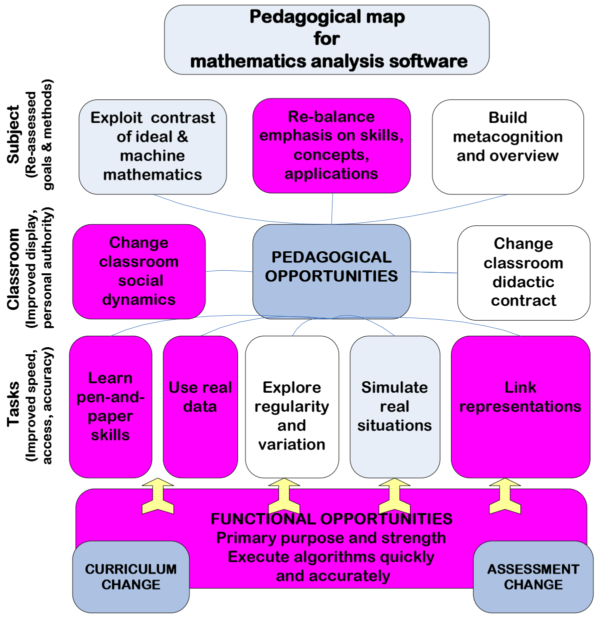

Fig. 7 Joe's use of MAS

MAS supporting enhanced engagement and deeper mathematical thinking

Ian, whose teaching is mapped in figure 8, was also teaching a middle secondary school class. He wished to enhance students' engagement through the regular use of group tasks, (i.e. change classroom social dynamics) set in real world contexts and also to deepen their mathematical thinking (Brown, 2005; Galbraith et al, 2007). He devised a series of investigations to supplement the teaching of every topic throughout the year which students were encouraged to work through independently from the teacher (i.e. changed didactic contract). Students were expected to progress the solution of problems by exploring different representations of functions. In addition Ian sometimes used dynamic geometry to show simulations of problems.

This teacher would also deliberately assign tasks which would produce unexpected technology output or at least output that would require the students to engage in further mathematical thinking not just automatic button pushing e.g. setting a suitable graph window or recognizing equivalent algebraic expressions. We classify these as simple instances of exploiting the limitations and anomalies of technology and showing the contrast between pen-and-paper and machine mathematics.

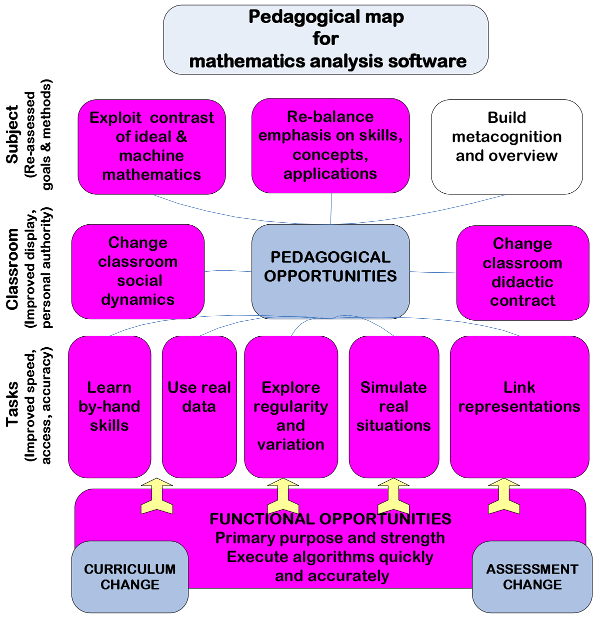

Fig. 8 Ian's use of MAS

Conclusions

In its many varieties and physical and electronic instantiations, mathematics analysis software has progressed a long way since the first hand held arithmetic calculators but Etlinger's comments still apply: to maximise the value for their students, teachers need to use MAS for both functional purposes and pedagogical purposes. The pedagogical map is intended to organise the types of opportunities that are available and thereby to provide a tool for researchers, teachers and educators involved in professional development. In the introduction of this paper, various uses for the pedagogical maps were presented, grouped as drawing attention to possibilities of teaching (e.g. in lesson planning or in teacher education) and as a tool to describe practice (e.g. for research, or for identifying needed professional growth). The goal of this paper has been to describe the pedagogical map in sufficient detail to enable others to use it, and to give one example of its use (in this case to describe teaching practices).

A key claim of this paper is that the pedagogical map is in fact a taxonomy of pedagogical opportunities for MAS software, and we are aware that giving examples and citing from limited literature cannot establish the comprehensiveness required to make a taxonomy. We have trialed the comprehensiveness of the map by considering the practice of various teachers who have been involved in previous research projects conducted by us or colleagues, including but not restricted to the examples given above. We have been able to map all aspects of their pedagogical use of MAS. Different researchers have independently created the same maps and or been able to identify a teacher, whose practice they are familiar with, from a given map. The test of its comprehensiveness is, however, whether other users find that its levels and categories, when broadly interpreted, can usefully describe pedagogical uses made in other parts of the world.

We also acknowledge that focusing on pedagogical opportunities does not highlight the pitfalls for introducing MAS. The map describes positive aspects of using MAS, but using MAS in mathematics classrooms can be problematic. Thomas, Monaghan and Pierce (2004), for example, note the widespread cautions with regard to the possible impact of CAS use on students' pen-and-paper algebra skills so that they become dependent on technology for too many computations; and the way in which learning to use CAS takes time that may previously have be devoted entirely to mathematics. These cautions are not part of the pedagogical map of opportunities, but they are important considerations for planning practice. MAS used fluently provides speed, accuracy and access but an overhead of learning time is paid to attain these.

Reflecting on the results of the mapping exercises presented above gives some indication of how the map behaves and how a map can be interpreted. For example, Lucy is a highly competent user of MAS and an excellent teacher, yet she chose to use MAS in a restricted way in the observed teaching setting throughout the school year. This is a map of that practice: in other settings we expect that she would have exploited additional pedagogical opportunities. It is not the case, therefore, that having more boxes shaded indicates better practice. Shading will depend on the teaching assignment but we do contend that a teacher who is aware of more pedagogical opportunities has stronger skills for meeting a variety of teaching needs. This is the key to the use of the pedagogical map for teacher development.

We were surprised to see the amount of variation in patterns on the map. The pedagogical opportunities that the teachers have exploited reflect their values in teaching mathematics and their perceptions of the principal needs of the group of students under consideration. The concern of teacher educators should be to ensure that teachers understand how MAS can be used in order to help achieve their goals.

MAS is developing rapidly in its functional capability and it is also becoming more readily accessible to more teachers and more classes in more settings. If teachers are to gain value from MAS then individual teachers must progress quickly in the complex process of instrumental genesis for teachers, referred to by Artigue (2005). To make good use of MAS, teachers have a lot to master, in becoming skilled users for functional purposes and also in understanding its pedagogical opportunities. To provide effective support for teachers in this complex process, it becomes important to understand what this development process looks like in practice and to describe good practices. We offer the pedagogical map for researchers, leaders of professional development, teacher educators and teachers to be used in this endeavour.

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