Assessing the Potential Suitability of “Show That” Questions in CAS-Permitted Examinations

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Examples of “show that” examination questions highlight the various ways CAS can impact upon question design, possible solution approaches, documented responses and marking schemes. A classification scheme used on some recent non-CAS NSW and Victorian examinations found that CAS and the role of automatic simplification can influence the solution process in a variety of ways. Assessment issues such as control over CAS and the nature of solution accounts are also discussed using student responses from a pilot CAS subject in Victoria.

The construction of examinations that allow students to use CAS pose direct *technical* and *pedagogical* challenges for external examination setting panels, and hence, for teachers setting assessment tasks and textbook authors. In a technical sense, CAS influences many facets of examination question design practices such as structure, language, explicitness, marking scheme protocols and equal access for users of different permitted machines. On a pedagogical level, careful consideration needs to be given to the nature and bandwidth of mathematical abilities we want (and are able) to assess in CAS-permitted examinations. In particular, consideration must be given to the viability of testing students’ abilities to prove mathematical results when CAS could be used to construct the proof. In this paper, the suitability of current non-CAS “show that/prove” examination questions for use in future CAS-permitted examinations are scrutinised by examining the impact CAS has on intermediate steps and the implications for question design, the nature of student responses and student control of CAS. A first example is used to demonstrate the need for scrutiny of “show that” questions and the problems that specification of form of results can cause for CAS users. A second example reports on students’ responses to a “show that” question designed to avoid common pitfalls. Finally, a survey of “show that” questions from recent NSW and Victorian examinations provides a catalogue and classification system of the many ways in which these questions can be affected by CAS.

Relevant Literature

Flynn & McCrae (2001) and Flynn (2001) categorised graphics calculator-permitted examination questions in Victoria according to the degree of CAS impact. These schemes classified questions as those where CAS has either (i) no impact; (ii) some impact but remaining useable as the question’s conceptual difficulty is preserved (although slight adjustments in structure, wording or marking schemes may be needed); or (iii) sufficient impact that the question would be omitted from a CAS-permitted examination because it only tests knowledge of a procedure that can be done with CAS. These schemes demonstrated that CAS use can render some traditional examination (algebra and calculus) questions redundant for either technical and/or pedagogical reasons and cause a shift in what mathematical knowledge a question is actually testing. Included in this subset were questions where CAS use eradicated all possible intermediate steps and hence a student

Flynn. P., & Asp, G. (2002). Assessing the potential suitability of "show that" questions in CAS-permitted examinations. In B. Barton, K.C. Irwin, M. Pfannkuck & M. O. J. Thomas (Eds.), Mathematics Education in the South Pacific (Proceedings of the 25th Annual Conference of the Mathematics Education Research Group of Australasia, pp. 252 - 259). Auckland: MERGA.

would be unable to provide a written solution account unless some intermediate steps were deliberately done by hand.

Several authors (for example Lagrange, 1999; Pierce & Stacey, 2001) outline other mathematical abilities that will increase in importance when using CAS. Students must possess algebraic insight including being able to convert conventional mathematics to the required syntax for entry into CAS and to recognise equivalent expressions when interpreting CAS outputs and checking the reasonableness of results (Pierce & Stacey, 2001). Algebraic insight is needed for doing mathematics by hand, but it is a greater part of work with CAS, partly because the CAS often throws up unexpected results that would never arise by hand. Lagrange argues that while traditional algebraic techniques will become less important, students using CAS will need to master new techniques to be able to reflect on and demonstrate algebraic equivalence of expressions generated either by-hand and/or by-CAS in addition to understanding the form of CAS outputs.

In addition to shifts in emphasis of required algebraic knowledge, shifts will also occur in how solution accounts are documented. Ball & Stacey's (in press) RIPA rubric suggests a framework for recording solutions with CAS, to help teachers ensure that students learn good mathematical practice for presentation of solutions assisted by technology. The framework stresses that the overall solution plan must be apparent and there must be clear recording of mathematical reasoning, inputs to CAS and selected CAS outputs following correct mathematical syntax and conventions. Hence, it is anticipated that whilst there will be a decreased emphasis on presenting manipulative intermediate steps, this needs to be compensated for by enhanced written evidence in choice of solution approach.

There are many instances in 1995-2000 Victorian VCE and New South Wales' HSC Examiners' Reports that comment upon current student difficulties in addressing non-CAS "show that" examination questions successfully. See, for example, these quotes firstly from the 1998 VCE Specialist Mathematics Report for Teachers (Board of Studies, VIC 1999) and secondly the 1998 NSW HSC Examination Report (Board of Studies, NSW 1999):

> "This year there were a number of questions that required students to establish results given on the paper. The answers to such 'show that' questions are often poorly presented and it needs to be stressed to students they must show, clearly and logically, every step that leads to the result". (p 22)

> "It is also worth emphasising that, in questions where the answer is given, it is the responsibility of the candidate to ensure that sufficient lines of working are given to convince the examiners that the result has been shown". (p 61)

Both examination bodies see that students currently find difficulty adequately demonstrating reasoning with by-hand solutions. Coupled with automatic simplification, the influence of CAS could have important implications for the structure of "show that" examination questions, the nature of student responses and their ability to control their CAS successfully.

"Show That" Examination Questions

Dual Functions in Examination Assessment

It is a relatively common practice to structure examination questions around instructions such as "show that" or "prove" (words which we believe have the same intention). These questions perform two important functions in examination design. One is to assess students' ability to construct and present a logical well-reasoned mathematical argument using a sequence of intermediate steps. These intermediate steps, rewarded through marking schemes, can incorporate the testing of many facets of mathematical knowledge including use of mathematical definitions, facts and theorems and employment of various techniques, procedures and algorithms. Through these questions, students can demonstrate their knowledge of the connections between mathematical ideas. For this reason, it seems important that "show that" questions be retained in new assessment.

A second function of "show that" questions is to increase accessibility by supplying a result required in later question parts. In addition to assisting students through these gateways, the supplied results are usually in a form which enhances the likelihood of student's successfully completing later parts of the question (e.g. in a form easy to integrate, interpret etc). This function of "show that" questions is therefore to make difficult questions easier by providing implicit guidance along the solution path.

A First Example

This section raises some of the issues with "show that" questions by discussing an example. Flynn & McCrae (2001) and Flynn (2001) found that CAS use can render some traditional examination (algebra and calculus) questions unusable for either technical and/or pedagogical reasons and cause a shift in what is tested in a question. A subset of these unsuitable questions can be completed by one direct input into CAS (without any formulation or abstraction), with CAS providing an identical output to the required result, meaning no intermediate steps can be documented. Such a situation occurs in the "show that" examination question from the 2000 International Baccalaureate Mathematical Methods Standard Level Paper 2 (IBO, 2000) shown in Figure 1. A student using a TI-89 would be unable to provide any account of their solution to part (a) because the required factored result is provided immediately as an output. Note this is not necessarily the case for all CAS, an issue which is taken up in the next section. Part (a) also loses its value in a CAS-permitted examination because it is no longer testing whether students can use the product rule of differentiation. This is a clear example of the potential influence of CAS to cause shifts in the mathematical abilities assessed.

The diagram (not reproduced here) shows part of the graph of the curve with equation

$$y = e^{2x} \cos x.$$

(a) *Show that* $\frac{dy}{dx} = e^{2x} (2 \cos x\text{-}\sin x)$.

(b) Find $\frac{d^2y}{dx^2}$.

(c) There is an inflexion point at P(a, b).Use the results from (a) and (b) to *prove that*

(i) tan a = 3/4 and

(ii) the gradient of the curve at P is e^{2a}

Figure 1. IBO (2000) Mathematical Methods Standard Level Paper 2, Question 7

Challenges in Setting "Show That" Examination Questions for Different CAS.

Interest in structuring fair questions for users of different CAS is precipitated by the introduction in Victoria of VCE Mathematical Methods (CAS) which permits different CAS in its examinations. Figure 2 below shows the potential difficulties in setting "show that" CAS-permitted questions for users of different CAS (labelled CAS 1 and CAS 2 in the figure). The solutions illustrating CAS input (I), CAS output (O) and non-CAS (N) address the proof question 7(c) (figure 1) which require use of results obtained or given in parts (a) and (b). A CAS 2 user needed to demonstrate a little more knowledge for parts (a) and (b) to obtain the factored forms, and greater knowledge of trigonometric identities than a CAS 1 user to show the results in part (c). A CAS 2 user may need greater flexibility moving between by-hand and CAS techniques due to interruptions caused by unexpected CAS output. The solutions provided for a ($a = n\pi + \tan^{-1}(3/4)$ and $a = 2k\pi + 2\tan^{-1}(1/3)$) are equally valid, but the form of the specified result privileges CAS 1 over CAS 2. A CAS 2 user can also show the last result by using the approximate function that operates less directly than CAS 1 to simplify the trigonometric expressions. This shows that different CAS can offer different solution pathways, intermediate results and can be mode sensitive.

CAS 1	CAS 2
(I) Differentiate(*expression, variable*)	(I) Differentiate(*expression, variable*)
(O) $\frac{dy}{dx} = e^{2x}(2\cos x - \sin x)$	(O) $\frac{dy}{dx} = 2e^{2x}\cos x - e^{2x}\sin x$
	(I) Factor(*Answer*)
	(O) $\frac{dy}{dx} = e^{2x}(2\cos x - \sin x)$
(I) Differentiate(*Answer, variable*)	(I) Differentiate(*Answer, variable*)
(O) $\frac{d^2y}{dx^2} = e^{2x}(3\cos x - 4\sin x)$	(O) $\frac{d^2y}{dx^2} = 3e^{2x}\cos x - 4e^{2x}\sin x$
	(I) Factor(*Answer*)
	(O) $\frac{d^2y}{dx^2} = e^{2x}(3\cos x - 4\sin x)$
(I) Solve(*equation, variable*)	(I) Solve(*equation, variable*)
Solve $e^{2a}(3\cos a - 4\sin a) = 0$ for a.	Solve $e^{2a}(3\cos a - 4\sin a) = 0$ for a.
(O) $a = n\pi + \tan^{-1}(3/4)$ where $n \in J$.	(O) $a = 2k\pi + 2\tan^{-1}(1/3)$ where $k \in J$.
	(N) $t = \tan(a/2)$ and $\tan a = 2t/(1-t^2)$, $\tan a = 3/4$.
(I) tan(*answer*)	
Apply the tan function *to both sides*	(I) Apply tan *to both sides* in *approx.* mode
(O) $\tan a = 3/4$.	(O) $\tan a = 3/4$.
(I) Substitute into derivative.	(I) Substitute into derivative. $f'(\tan^{-1}(3/4))$ is $e^{2\tan^{-1}(3/4)}(2\cos(\tan^{-1}(3/4)) - \sin(\tan^{-1}(3/4)))$.
(O) $f'(\tan^{-1}(3/4)) = e^{2\tan^{-1}(3/4)}$ which is e^{2a} as $a = \tan^{-1}(3/4)$.	(I) $2\cos(\tan^{-1}(3/4)) - \sin(\tan^{-1}(3/4)) = 1$, Hence $f'(\tan^{-1}(3/4))$ is e^{2a} in terms of a.

Figure 2. Solutions to IBO 2000 Question 7, using two different CAS.

These sample solution records illustrate that “show that” questions can be problematic because calculator outputs are dependent upon the internal simplification code and the capabilities of the particular CAS. On the other hand, if intermediate answers are not specified, the gateways to direct solution paths are absent so the question would become more difficult. This example shows that challenges will confront students when using CAS in examination assessment. Students will be required to present logical well-reasoned solution accounts when using CAS and they must be able to exert a strong degree of *control* over their CAS in these situations. This control must be maintained throughout any input/output interaction with CAS. The predisposition of CAS to simplify expressions automatically and its sensitivity towards syntax of entered expressions could present difficulties for students controlling and responding to CAS inputs and outputs.

Example of a “Show That” CAS-Permitted Examination Question

Setting/Methodology

We wanted to investigate the viability of setting “show that” examination questions influenced by CAS and to commence looking at whether students could (i) employ the capabilities of CAS with enough control to address such questions successfully; (ii) present logical and sufficiently detailed solutions; and (iii), move flexibly between by-hand and CAS techniques. One of the three participating schools in The University of Melbourne’s CAS-CAT project (http://www.edfac.unimelb.edu.au/DSME/CAS-CAT) used a “show that” examination question (Figure 3 below) in their internally reported 2001 Unit 1 VCE Mathematical Methods (CAS) Examination. Forty two Year 11 students sat the examination and were permitted to use a Casio FX 2.0. The students had five months experience using CAS and hence were still developing strategies for effective CAS use. CAS impacts upon this question by decreasing the number and nature of intermediate steps and provides a natural solution approach because of the CAS’s facility to factor the expression directly. Alternatively, CAS could be used to expand and then factor the expression. The marking scheme awarded one mark for a correct substitution; one method mark for obtaining $(x^2 + 3y^2)^3$ or a partially factored variant of this and a third mark for explicitly relating the factored result to being a perfect cube. The third mark was only awarded if the obvious perfect cube form was specified. The analysis below is based only on the written records of the students. Each student’s solution was assessed in accordance with the marking scheme, various solution approaches categorised and interesting features in solution accounts noted.

Let $a = x^3 - 9xy^2$ and $b = 3x^2y - 3y^3$. Show that $(a^2 + 3b^2)$ is a perfect cube.

Figure 3. Unit 1 VCE Mathematical Methods (CAS) Examination Question

Results/Implications for CAS-Permitted Examination Assessment

Table 1 below describes the approaches used and a measure of their success. Despite five months exposure to CAS, a significant proportion of students (22/42) did not use CAS. Students using by-hand methods had to undertake lengthy and highly error-prone approaches to obtain the perfect cube form and, not surprisingly, were not successful, with familiar errors in substitution, expansion and factorisation exhibited. Twelve out of 18 students who used CAS were successful in establishing the perfect cube result.

Table 1
Profile of Student Responses to 'Show That' Examination Question

Response Mode	Not Attempted	No CAS	CAS	Mixture
Responses	3	19	18	2
Correct Answers	0	1	12	0
Mean score out of 3	0.0	0.8	2.2	1.5

These students typically demonstrated the result by explicit recording of the mathematical operations (eg using the words factor, simplify or substitute) rather than making use of these mathematical operations implicitly in manipulative intermediate steps such is done in non-CAS situations. The successful CAS students invariably informed the marker of their solution approach, the mathematical operations used and they recorded selected outputs. This is shown in Figure 4 below, illustrating features of CAS-style "show that" responses recorded by two students.

Few students amalgamated CAS and by-hand techniques. This weak interplay was signified by students almost universally selecting either CAS or non-CAS at the start of the solution process and continuing with one approach. There was little evidence of students moving flexibly between non-CAS and CAS approaches even after experiencing algebraic difficulties with the former.

Student A	Student B
$(x^3 - 9xy^2)^2+3\ (3x^2y - 3y^3)^2$	*Factor* $(x^3 - 9xy^2)^2+3\ (3x^2y - 3y^3)^2$
Factor gives $(x^2 + 3y^2)^3$	*Factor* $x^2(x\text{-}3y)^2(x+3y)^2+3(3x^2y\text{-}3y^3)^2$
This is in perfect cube form	$(x^2 + 3y^2)^3$ (The power 3 means a perfect cube).

Figure 4. Sample Student CAS Responses to "Show That" Question

A tension was noticed amongst some students with inputting the correct algebraic expressions accompanied by correct syntax, detecting the presence of syntactical errors and having the facility to deal with unexpected CAS outputs. Five students omitted an important set of surrounding brackets and entered *Factor (expression* instead of correctly entering *Factor ((expression)*. (Note that their calculator does not require a final closing bracket.) This resulted in a partial factorisation of the first part of the expression but the second part was left untouched. Two of these five students (e.g. Student B in the figure above) exhibited excellent algebraic expectation (Pierce & Stacey, 2001) by considering the output and again factoring to obtain the perfect cube form. The other three students confronted with this situation did not deal with this unexpected output and left the partially factored result as is, hence not obtaining the perfect cube form. Generally, students using CAS were reasonably successful with their control of CAS and were able to demonstrate the required result with well-reasoned solutions. In attempting questions in this manner, students must become accustomed to checking their inputs (syntax) and outputs for any unexpected results.

Recent non-CAS "Show That" Examination Questions

Analysis of the "show that" examination question described in the previous section highlights the possible decrease in required intermediate steps and greater emphasis placed on explicit naming of mathematical operations in solution accounts. This encouraged us to look at the effect of CAS on "show that" examination questions more widely. A classification scheme adapted from previous schemes employed by Flynn & McCrae (2001) and Flynn (2001) was used to categorise 'show that' examination questions from the 1997 to 2001 VCE Specialist Mathematics Examination 2 and the 1997 and 2000 NSW HSC 3 Unit (Additional) and 3/ 4 Unit (Common) Examinations. Each question was answered using a TI-89 and classified according to the *degree that CAS impacted upon the number and nature of required intermediate steps*. The percentage of marks of "show that" questions affected was noted and are shown in Table 2. The table illustrates the marks allocated to "show that" questions as a percentage of the total marks available in the examination and the distribution of the three categorised types of "show that" questions expressed as a percentage of the total marks allocated to "show that" questions. The last column in the table provides the percentage of marks allocated to "show that" question parts where the result was needed for a later question part.

Table 2
Distribution of marks allocated to "Show That" Examination Questions by Type

Examination	Total Marks (%)	Types			Result Needed (%)
		1	2	3	
(1997-2001) VCE Specialist Mathematics Exam 2	17	23	65	12	85
1997 HSC 3 Unit (Additional) and 3/4 Unit (Common)	32	19	74	7	67
2000 HSC 3 Unit (Additional) and 3/4 Unit (Common)	33	25	75	0	46

Type 1 questions contain no CAS-affected intermediate steps. They typically include geometry and other questions not involving algebra; formulation/abstraction of mathematical expressions from worded or diagrammatic information; or interpretation of properties of unspecified functions. Type 3 "show that" examination questions are unsuitable in CAS-permitted examinations because using CAS devours all the intermediate steps and provides the desired result without giving students the need to display any intermediate mathematical reasoning. Part (a) of the question in Figure 1 is an example for a TI-89 user. For CAS-permitted papers, these questions need to become "find" type questions.

Type 2 questions are somewhat affected by CAS. These may still be useable in CAS-permitted examinations, with modification, but the emphasis of what is being tested may change. Some Type 2 questions would not be asked in a CAS-permitted paper as these results are provided currently to assist students answering questions that otherwise could not be done, for example, the direct technique is outside the syllabus. CAS would influence these questions to become "find" style questions. In some cases, solution pathways are parallel, where CAS use compresses the number of required intermediate steps (e.g.

question in Figure 3). In others cases, the automatic simplification of CAS takes you on a different solution pathway (e.g. Figure 1 part (c) for CAS 2 user) or it forces the interplay between CAS and by-hand techniques. There are also occasions where CAS can be used in a by-hand like manner to construct the required solution. There are many variants belonging to this type, and space does not permit the variety to be demonstrated here.

Conclusion

The examples discussed here and the analysis of selected non-CAS examination papers demonstrate the care needed when setting CAS-affected "show that" questions in CAS-permitted assessment and the variety of ways CAS can impact upon them. "Show that" CAS-permitted examination questions should have their suitability scrutinised in terms of the impact of CAS (including different CAS) on question structure and wording, intermediate steps, anticipated solution responses and marking scheme frameworks. These questions can be strongly affected by CAS use, particularly when construction of the proof is achieved mainly by symbolic manipulation and/or use of algebraic or trigonometric identities. Automatic simplification, differences in brand capabilities and internal simplification codes between various CAS and appropriate CAS settings can also affect responses. However, replacement of 'show that' examination questions with 'find' type questions in CAS-permitted examinations could unduly raise the overall level of difficulty and lower the accessibility of examinations.

Retaining "show that" examination questions is important because they are one pointer to the central role that proof plays in mathematics. Demonstrating how concepts are linked and displaying a logical argument is possible in CAS-permitted assessment, but achieving it fairly will require some imaginative recasting of current question types.

Acknowledgments

This work has been funded by a grant from the Australian Research Council with industry partners the VCAA, Texas Instruments (Australia), Shriro (Casio) and Hewlett-Packard (Australia). Thanks to Kaye Stacey for discussion and critical reviewing.

References

Ball, L., & Stacey, K. (in press). What should students record when solving problems with CAS? Reasons, information, the plan and some answers. In *Computer Algebra Systems in School Mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Board of Studies NSW, (1999). *1998 NSW HSC Examination Report*. Board of Studies. Sydney: Author.

Board of Studies, (1999). *1998 Report for Teachers*. Board of Studies. Melbourne: Author.

Flynn, P. (2001). Examining mathematics with CAS: Issues and possibilities. In C. Vale, J. Horwood & J. Roumeliotis (Eds.), *2001 A Mathematical Odyssey*. (pp 285- 298) Melbourne: MAV

Flynn, P., & McCrae, B. (2001). Issues in assessing the impact of CAS on mathematics examinations. In J. Bobis, B. Perry & M. Mitchelmore (Eds.), *Numeracy and Beyond* (Proc. of the 24th Annual Conference of the Mathematics Education Research Group of Australasia, (pp. 210-217). Sydney: MERGA.

International Baccalaureate Organisation, (2000). *Mathematical Methods Standard Level Paper 2*. Geneva: Author.

Lagrange, J-B (1999). Complex calculators in the classroom: theoretical and practical reflections on teaching pre-calculus. *International Journal of Computers for Mathematical Learning*, 4(1), 51-81.

Pierce, R. & Stacey, K. (2001) A framework for algebraic insight. In J. Bobis, B. Perry, M. Mitchelmore (Eds.), *Numeracy and Beyond (Proceedings of the 24th Annual Conference of the Mathematics Education Research Group of Australasia.* (Vol 2) (pp. 418-425). Sydney: MERGA.