# Solving equations: Will a more general approach be possible with CAS?

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Increased access to CAS will impact on the type of learning that will occur in mathematics classrooms in the future. In considering a course where CAS calculators are available at all times the approach to solving equations will most likely change. One outcome is that students will have the opportunity to deal with general functions on a more regular basis. To be able to solve equations, students will need to develop the ability to use algebraic manipulation to rewrite equations in a form where they can use standard techniques to find a solution.

## Solving equations with CAS

Access to CAS allows for a reconsideration of approaches to solving equations with students. When using CAS to solve equations there are some questions where the answer can be obtained in one step using the solve feature however there are others where this is not the case. Students need to develop general strategies to solve equations including skills to deal with equations that CAS may not be able to solve immediately. Equations that cannot be solved in one step require the selection of an appropriate mathematical technique, which is often algebraic, to find a solution for a problem. These general approaches are useful for students understanding of equation solving, even for problems where CAS provides immediate answers. It is probable that the use of CAS may enable students to develop a more algebraic approach to solving equations.

## CAS use requires good algebraic understanding

In the CAS-CAT research project (http://www.edfac.unimelb.edu.au/DSME/CAS-CAT) we have considered approaches for solving equations when students have access to CAS calculators for all mathematical work, including assessment. Development of teaching material for solving equations involved consideration of the skills students require when CAS performs routine calculations formerly done by hand. Reflection on the skills and mathematical understanding needed to solve equations suggests that what is important may be different when CAS is readily available. The possibilities for teaching and learning are certainly different and a more generalised development of an approach to solving equations is possible.

In attempting a range of problems using CAS it has become increasingly obvious that good algebraic understanding is essential to solve equations both without, but especially with, access to CAS. The extent to which CAS assists in solving equations depends largely on the equation. For simple linear equations, the CAS calculator is always able to solve the equation quickly, but so will most students using pen and paper. Beyond simple linear equations a CAS does not always give a result that is easily recognisable by students, as it will often switch into complex mode or the particular mode setting may result in an unexpected output. Another possibility is that a CAS is not able to find a solution in one step. In these cases, students need considerable algebraic understanding to transform equations into a form where the CAS can assist in finding the answer.

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Consider the equation $\frac{pw}{\sqrt{p^2+100}}-1=0$ that formed part of the solution to a question on a year twelve examination (Board of Studies, 1998) in Victoria. To find the solution to this equation, students had to solve for *p*. Initially students should attempt to solve this equation for *p* using the solve feature on a CAS. It is worth noting that not all CAS can solve this equation directly for *p*. If the CAS is not able to produce a solution immediately, students will need to find a number of equivalent equations in order to be able to rewrite in a form where *p* is the subject. When solving this equation, students could start by rewriting the equation as $pw=\sqrt{p^2+100}$. A good first step here is to write the equation without fractions. This is a useful technique for students, particularly in attempting to rewrite in a form where solutions can be produced using standard techniques. Students should then recognise that the resultant equation is easier to deal with by squaring both sides and considering powers of two rather than non-integer powers. This gives $p^2w^2-p^2=100$ and then $p^2=\frac{100}{w^2-1}$ where $w^2 > 1$, resulting in $p=\pm\sqrt{\frac{100}{w^2-1}}$. Students then had to interpret the result in the context of the original question. This question required students to have a good sense of solving equations. After obtaining $pw=\sqrt{p^2+100}$ an approach was required that enabled isolation of a power of *p*, in this case $p^2$. This example was not straightforward for all CAS. With some CAS students almost had to carry out the same procedure as if using a 'by-hand' method. For some CAS, the steps stages in solving this equation were:

1. Try to use the inbuilt solve feature of the CAS. If this works, write down the answer and interpret this in the context of the original problem.
2. If the inbuilt solve feature does not give an answer immediately, then students will need to rewrite the equation to isolate $p^2$. At each stage of working, try to use the solve feature of the CAS. Rewriting could involve factorisation, simplification and use of inverse relations.

To solve this question students require a level of algebraic understanding, even when CAS is readily available. The example shown above demonstrates the importance of basic algebraic manipulation in solving some questions where CAS may be unable to produce an answer. A unit on solving equations needs to help students continue to develop this basic algebraic manipulation and the ability to produce a range of equivalent equations for a given equation.

## The form of an equation is a key element in solving equations

Students need to develop an understanding of approaches for solving equations that are applicable in a variety of contexts and for a range of different functions. The following examples will attempt to illustrate some algebraic skills that are useful for developing approaches to solve a variety of equations. A benefit of using CAS is the opportunity to explore more functions with students and to overcome simple errors made by students when using by hand techniques.

A key element in being able to solve equations seems to be the ability to recognise the form of an equation (see for example, Pierce & Stacey, 2001). Students should to be able to recognise an equation as linear, quadratic, cubic or as another standard type of equation. Following identification of the type of equation, algebraic manipulation could be used to rewrite the equation in a form where known techniques can then be used to solve the equation.

An essential skill for students is the ability to recognise equivalent equations and use algebraic manipulation to produce a range of equivalent equations. For example,

$4\sin x + 6 + 2\sin^2 x = 4$ is a quadratic equation in sin $x$ and an equivalent equation is $\sin^2 x + 2\sin x + 1 = 0$. This ability to produce equivalent equations enables students to rewrite equations in a form where they can recognise the type of equation they are solving and hence the nature of the solution expected. This is particularly useful when the number of solutions may not be apparent to students from the initial equation that they are dealing with. Whenever possible students need to understand what they are expecting for an answer particularly when different CAS present answers in different ways.

The next example shows how we can help students develop an understanding of the form of an equation. This illustrates a general way of considering equations and provides an approach for solving equations that will be useful if students are solving equations that are unable to be solved directly using a CAS. The ability to recognise the form of an equation will be useful in developing a general understanding of solving equations.

In the following examples, *f(x)* will be taken to represent basic power functions $x^n$ and basic transcendental functions such as $e^x$, *sin x, cos x, tan x* and $\log_a x$.

Consider the equation $f(x) = \dfrac{4}{f(x) - 3}$ where students are required to solve for *x*. In dealing with equations that involve fractions, the first step would be to rewrite this equation in a standard form as $f(x)\left(f(x) - 3\right) = 4$ and then $\left[f(x)\right]^2 - 3f(x) = 4$. Students should recognise that $\left[f(x)\right]^2 - 3f(x) = 4$ is a quadratic equation in *f(x)* and that this can be rewritten as $\left(f(x) - 4\right)\left(f(x) + 1\right) = 0$. This results in $f(x) - 4 = 0$ and $f(x) + 1 = 0$ giving $f(x) = 4$ and $f(x) = -1$. These equations are solvable for *x* if the inverse is available, possibly over a restricted domain. For example if $f(x) = e^x$ then $f^{-1}(x)$ will give the solution, whereas, if $f(x) = \sin x$ then the family of solutions can be found using arsin *x*. Depending on the basic function, students will need to state any restrictions on the values that arise through rewriting equations. They will also need to interpret the solution obtained in the context of the original problem. The emphasis here is on recognising the form of the equation, in this case as a quadratic in *f*(*x*), and then standard techniques enable the solution to be determined. For the problem just considered, essentially there are six stages in solving the equation:

1. Rewrite the equation without fractions.
2. Recognise the form of the equation $\left[f(x)\right]^2 - 3f(x) - 4 = 0$ as a quadratic.
3. Use a standard technique, in this case factorisation, to solve the equation for *f(x):* $\left(f(x) - 4\right)\left(f(x) + 1\right) = 0$
4. Use the inverse relation to write down the solution, $x = f^{-1}(4)$ and $x = f^{-1}(-1)$ if the inverse exists or use special techniques for trigonometric functions.
5. Write down a solution for the problem where the inverse is not uniquely defined. For example, $x^2 = 4$ so $x = \pm\sqrt{4}$. If $f(x) = \sin x$ then $f(x) = -1$ has a family of solutions.
6. Check the answer in the context of the original problem.

Students could use this approach for any basic function *f(x)*, so the techniques used to solve this equation are applicable for a range of problems. The approach where students identify the form of an equation is useful in the solution of many equations. Clearly, when students have CAS they can try to use the inbuilt solve feature whenever possible however an understanding about solving equations is important.

## A general approach to solving equations

An approach to help students develop their skills at solving equations could be to consider equations of a general form and then move to specific examples, rather than consider many specific examples and then generalise the results obtained. For example, if students start with an equation such as $a = \frac{b}{f(x)} + d$ they could find *f(x)* first before writing down an equation to solve for *x*. This result could be used to find *f(x)* for a range of different values of *a, b* and *c* and for a variety of basic functions. This type of simple algebraic manipulation will be important even when CAS is available. Similarly, for an equation of the form $a\,f(x) + b = c$ students can write *f(x)* in terms of *a, b* and *c,* namely $f(x) = \frac{c-b}{a}$. They could then consider what the solution will be for a range of different basic functions such as $f(x)= x^2$ or $f(x) = \sin x$. *f(x)* can be stated for each case using the general result. Students need to recognise that any equation of the standard form $a\,f(x) + b = c$ can be rewritten to give *f(x)*. This approach can be used to solve equations of the form $4f(x) + 5 = 7$ where *a*, *b* and c have been allocated specific values. Students could consider examples such as $4x^4 + 5 = 7$, $4x^{0.5} + 5 = 7$ or $4\cos x + 5 = 7$ and write equations for *f(x)* in each case. Solving any equation of this form gives $f(x) = 0.5$ and hence $x = f^{-1}(0.5)$ where the inverse exists. A similar approach could be used to solve equations of the form $f(ax+b) = c$ for *x*.

There needs to be an emphasis on how there may be restrictions on the values that *x* may take. For example, when solving and either squaring or taking the square root this may affect the solutions that are possible. The solutions that students provide for problems should reflect this understanding of the restrictions on possible answers. Although CAS will solve most equations, it may still be important for students to have some understanding of a general approach for solving.

For students in their final years of secondary schooling there will be a number of algebraic skills requiring familiarity. In order to be able to get equations into a form where the solve function of a CAS can be used students may need to be able to employ some techniques that they would use to solve equations by hand. Being able to recognise equivalent equations will be useful for approaching questions where CAS may not provide an answer immediately and in this situation students may also need to complete part of the solution using a method similar to a by-hand method.

## What would we expect with CAS? – some initial thoughts

We should be encouraging students with access to CAS to attempt to solve most given equations using the inbuilt solve feature. Exceptions would be cases where it may be more efficient for a student to simply write down an answer rather than enter an equation into a CAS, or where an answer may be immediately apparent. Examples could include the solution of a linear equation or an equation such as $e^x = 2$ for *x*.

Students should be aware that it is not always possible to obtain an answer in one step using the solve feature of a CAS calculator and that at this stage there are limitations to the equations that CAS will solve. To use CAS students may still need to rewrite an equation so that it has a more standard form and this will require algebraic skills. Students may need to carry out intermediate steps in the process of finding the answer for an equation when the CAS cannot immediately produce answers. If a CAS cannot solve an equation in one step then this ability to rewrite an equation may assist students in producing a standard form that the CAS can solve. In a unit on solving equations there would be an expectation that students would develop a general sense of the connections between various algebraic expressions or equations and how to produce equivalent forms.

Once students have obtained an answer using CAS they will need to write down the answer to the given problem in the context of the original problem. Many problems will require students to interpret results after obtaining a solution. For example, students may be required to substitute values for a parameter to find a solution within a restricted domain and then comment on the results in the context of the problem. An example is in stating the solution for the equation $2\,sin^2\,x - sin\,x - 1 = 0$ where $0 \le x \le 2\pi$ . After an output has been produced using CAS, students will need to find the answer in the restricted domain by substituting values for the parameter.

The implication of this more general approach to solving equations being possible is that the type of question we can ask of students can change. There is the scope to ask more questions that probe student understanding and that require interpretation of results. This opportunity to ask general questions enables us to explore students' algebraic understanding. For example, a question may require students to write down an equation with a specified number of solutions if $af(x) + b = 0$, rather than give a specific question with stated values for $a$, $b$ and $c$. This type of question would require students to choose a solution to an equation that satisfies particular requirements. To do this requires a good understanding of equations and the type of solution that can result from different functions, $f(x)$. CAS allows students to explore more functions, tackle more interesting questions and use a more general approach to solving equations.

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